Solutions to Problem Sheet 9

1. (a) Start with

$$I = \int_0^2 e^{-x^2}\,dx, \qquad n = 4.$$

Then for this problem, $f(x) = e^{-x^2}$ and

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}.$$

Consequently,

$$\begin{aligned} x_1^* = &\ \ 1/4, \qquad f(x_1) &= e^{-\frac{1}{16}} \\ x_2^* = &\ \ 3/4, \qquad f(x_2) &= e^{-\frac{9}{16}} \\ x_3^* = &\ \ 5/4, \qquad f(x_3) &= e^{-\frac{25}{16}} \\ x_4^* = &\ \ 7/4, \qquad f(x_4) &= e^{-\frac{49}{16}} \end{aligned}$$

Thus the estimate gained by using the Rectangular Rule is

$$\begin{aligned} I &\approx \frac{1}{2}\left\{e^{-\frac{1}{16}} + e^{-\frac{9}{16}} + e^{-\frac{25}{16}} + e^{-\frac{49}{16}}\right\} \\ &= 0.8828 \quad \text{(4 d.p.)} \end{aligned}$$

(b) i. Start with

$$I = \int_0^2 e^{-x^2}\,dx, \qquad n = 4.$$

Then for this problem, $f(x) = e^{-x^2}$ and

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}.$$

Consequently,

$$\begin{aligned} x_0 &= 0, & f(x_0) &= 1 \\ x_1 &= 1/2, & f(x_1) &= e^{-\frac{1}{4}} \\ x_2 &= 1, & f(x_2) &= e^{-1} \\ x_3 &= 3/2, & f(x_3) &= e^{-\frac{9}{4}} \\ x_4 &= 2, & f(x_4) &= e^{-4} \end{aligned}$$

Thus the estimate gained by using the Trapezium Rule is

$$\begin{aligned} I &\approx \frac{1}{4}\left\{1 + e^{-4} + 2\left[e^{-\frac{1}{4}} + e^{-1} + e^{-\frac{9}{4}}\right]\right\} \\ &= 0.8806 \quad \text{(4 d.p.)} \end{aligned}$$

ii. We can let

$$M = \frac{4}{e^{\frac{3}{2}}}.$$

Then the error estimate is

$$|\varepsilon_T| \leq \frac{4}{e^{\frac{3}{2}}} \frac{(2-0)^3}{12 \times 4^2} = \frac{1}{6e^{\frac{3}{2}}} = 0.0372,$$

or 3.72×10^{-2}, to three significant figures.

(c) i. For this problem,

$$I = \int_0^2 e^{-x^2}\, dx, \qquad n = 4.$$

Then

$$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2},$$

and (again),

$$\begin{aligned} x_0 &= 0, & f(x_0) &= 1 \\ x_1 &= 1/2, & f(x_1) &= e^{-\frac{1}{4}} \\ x_2 &= 1, & f(x_2) &= e^{-1} \\ x_3 &= 3/2, & f(x_3) &= e^{-\frac{9}{4}} \\ x_4 &= 2, & f(x_4) &= e^{-4} \end{aligned}$$

Next, compute

$$\begin{aligned} S_0 &= f(x_0) + f(x_4) = 1 + e^{-4} \\ S_1 &= f(x_1) + f(x_3) = e^{-\frac{1}{4}} + e^{-\frac{9}{4}} \\ S_2 &= f(x_2) = e^{-1}. \end{aligned}$$

Hence by Simpson's Rule, the estimate for the integral is

$$\begin{aligned} I &\approx \frac{h}{3}(S_0 + 4S_1 + 2S_2) \\ &= 0.8818 \quad (4 \text{ d.p.}) \end{aligned}$$

ii. This time, we can set $M = 12$. Then

$$|\varepsilon_S| \leq \frac{12 \times (2-0)^5}{180 \times 4^4} = \frac{1}{120} = 0.00833,$$

or 8.33×10^{-3}, to three significant figures.

2. We can label the three professors as A, B and c, and their corresponding coats as a, b and c respectively and consider the six possible outcomes...

A	B	C	Number correct
a	b	c	3
a	b	c	1
b	c	a	0
b	a	c	1
c	b	a	1
c	a	b	0

As each outcome is equally likely, the probability distribution for the random variable

$$X = \text{Number of properly dressed professors}$$

is as follows:

x	$P(X = x)$
0	$\frac{2}{6}$
1	$\frac{3}{6}$
2	0
3	$\frac{1}{6}$

Moreover, the expected number is

$$0 \times \frac{2}{6} + 1 \times \frac{3}{6} + 2 \times 0 + 3 \times \frac{1}{6} = 1.$$

3. This problem can be modelled using the distribution $B(16, \frac{1}{2})$. Define the random variable

$$X = \text{Number of green baubles},$$

then

$$\begin{aligned} P(X \leq 2) &= 1\left(\frac{1}{2}\right)^{16} + 16\left(\frac{1}{2}\right)^{16} + 120\left(\frac{1}{2}\right)^{16} \\ &= 137 \times \left(\frac{1}{2}\right)^{16} \\ &\approx 0.0021 \end{aligned}$$

In the second case, the new distribution is $B(16, \frac{1}{4})$, since now only one quarter of the baubles are green. This time,

$$\begin{aligned} P(X \leq 2) &= 1\left(\frac{3}{4}\right)^{16} + 16\left(\frac{3}{4}\right)^{15}\left(\frac{1}{4}\right) + 120\left(\frac{3}{4}\right)^{14}\left(\frac{1}{4}\right)^{2} \\ &\approx 0.197. \end{aligned}$$

4. This problem follows the binomial distribution $B(500, \frac{1}{100})$, but it will be approximated by $Po(\mu)$, with

$$\mu = 500 \times \frac{1}{100} = 5.$$

Let

$$X = \text{Number of defective lights},$$

then

$$\begin{aligned} P(X \leq 3) &= e^{-5} + \frac{e^{-5}\,5^1}{1!} + \frac{e^{-5}\,5^2}{2!} + \frac{e^{-5}\,5^3}{3!} \\ &= e^{-5}(1 + 5 + \frac{25}{2} + \frac{125}{6}) \\ &\approx 0.265. \end{aligned}$$