Solutions to Problem Sheet 8

1. (a) Because $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all unit vectors, we know that $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$. Then, using the definition of the dot product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta, \tag{1}$$

we have

$$\mathbf{i} \cdot \mathbf{i} = |\mathbf{i}||\mathbf{i}|\cos 0 = 1 \cdot 1 \cdot 1 = 1.$$

Similarly, we yield $\mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{k} \cdot \mathbf{k} = 1$.
Next, we apply (1) again. Using the observation that $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are mutually perpendicular, and $\cos\frac{\pi}{2} = 0$, we obtain the other six results, which are

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0, \qquad \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0, \qquad \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0.$$

Therefore all nine results have been proven.

(b) We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = & \quad (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ = & \quad a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ + & \quad a_2\mathbf{j} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ + & \quad a_3\mathbf{k} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}), \end{aligned}$$

but according to the results from part (a), this leads to

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = & \quad a_1b_1 + \quad 0 + \quad 0 \\ + & \quad 0 + \quad a_2b_2 + \quad 0 \\ + & \quad 0 + \quad 0 + \quad a_3b_3, \end{aligned}$$

hence we conclude that

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

2. Note that the diagonals of the rhombus start end end at opposite corners of the rhombus. Thus the diagonals are

$$\begin{aligned} \mathbf{c} &= \mathbf{b} + \mathbf{a} \\ \mathbf{d} &= \mathbf{b} - \mathbf{a}, \end{aligned}$$

and their dot product is

$$\begin{aligned} \mathbf{c} \cdot \mathbf{d} &= (\mathbf{b} + \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - \cancel{\mathbf{b} \cdot \mathbf{a}} + \cancel{\mathbf{a} \cdot \mathbf{b}} - \mathbf{a} \cdot \mathbf{a} \\ &= |\mathbf{b}|^2 - |\mathbf{a}|^2 \end{aligned}$$

However, because the shape is a rhombus, all four sides are the same length, so $|\mathbf{a}| = |\mathbf{b}|$ holds. Hence

$$\mathbf{c} \cdot \mathbf{d} = \cancel{|\mathbf{b}|^2} - \cancel{|\mathbf{a}|^2} = 0,$$

indicating that the diagonals of the rhombus are indeed perpendicular.

3. (a) Start by calculating $\overrightarrow{PQ}$ and $\overrightarrow{PR}$:

$$\begin{aligned} \overrightarrow{PQ} &= (2\mathbf{i} - \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= \mathbf{i} + \mathbf{j} - 3\mathbf{k}, \\ \overrightarrow{PR} &= (2\mathbf{j} + \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \\ &= -\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \end{aligned}$$

Use the vector product to compute the area, because

$$\text{Area of triangle} = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|.$$

So we calculate the vector product:

$$\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix} \\
&= -\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} - (-9\mathbf{i}) - (-\mathbf{j}) - (-\mathbf{k}) \\
&= 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}.
\end{aligned}$$

And so

$$\begin{aligned}
\text{Area} &= \frac{1}{2}|8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}| \\
&= \frac{1}{2}\sqrt{8^2 + 4^2 + 4^2} \\
&= \frac{\sqrt{96}}{2} \\
&\approx 4.899.
\end{aligned}$$

(b) Recall that the length of the vector product satisfies

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\overrightarrow{PQ}||\overrightarrow{PR}| \sin\theta, \tag{2}$$

and we know that

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}| = \sqrt{96}.$$

Meanwhile,

$$\begin{aligned}
|\overrightarrow{PQ}| &= \sqrt{1^2 + 1^2 + (-3)^2} = \sqrt{11} \\
|\overrightarrow{PR}| &= \sqrt{(-1)^2 + 3^2 + (-1)^2} = \sqrt{11}.
\end{aligned}$$

Hence putting all this into (2) gives

$$\sqrt{96} = \sqrt{11}\sqrt{11}\sin\theta,$$

which rearranges to

$$\sin\theta = \frac{\sqrt{96}}{11},$$

and so

$$\theta = \sin^{-1}\left(\frac{\sqrt{96}}{11}\right) = 62.96^\circ \text{ or } 1.0989 \text{ radians},$$

(c) We observe that the vector product, i.e.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}.$$

is perpendicular to both $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. However, it is not a unit vector because

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{96} \neq 1.$$

Fortunately we can obtain a unit vector out of $(\overrightarrow{PQ} \times \overrightarrow{PR})$ by using the following trick:

$$\hat{\mathbf{n}} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{1}{\sqrt{96}}(8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}).$$

Then $\hat{\mathbf{n}}$ is the desired unit vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Note: A similar method using $\mathbf{b} \times \mathbf{a}$ instead would still give a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. The only difference is that the new unit vector would point in the opposite direction, which would be indicated by a minus sign.