Solutions to Problem Sheet 7

1. (a) Start by using separation of variables:

$$\int \cos y \, \mathrm{d}y = \int \sin x \, \mathrm{d}x \qquad \Rightarrow \qquad \sin y = -\cos x + C,$$

where C is a constant of integration.
When $x = 0$, $y = \pi$, hence

$$\begin{aligned} \sin \pi &= -\cos 0 + C \\ 0 &= -1 + C \\ C &= 1. \end{aligned}$$

Therefore we have

$$\sin y = 1 - \cos x,$$

and we can now rearrange to make y the subject. We end up with

$$y = \sin^{-1}(1 - \cos x).$$

This is the general solution.

(b) This ODE turns out to be first-order and linear, but we need to get it in the right form first! This is done by dividing both sides by x, which gives

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^3}. \tag{1}$$

Observe that

$$P(x) = \frac{3}{x}, \qquad Q(x) = \frac{\sin x}{x^3},$$

therefore the integrating factor (IF) is

$$I(x) = e^{\int \frac{3}{x}\,\mathrm{d}x} = e^{3\ln x} = e^{\ln(x^3)} = x^3.$$

Now the ODE can be written in the form

$$\frac{d}{dt}(Iy) = QI,$$

by multiplying both sides of (1) by $I(x) = x^3$. We have

$$x^3\frac{dy}{dx} + 3x^2y = \sin x,$$

so the LHS is an exact type of the form

$$\frac{d}{dt}(x^3y) = \sin x.$$

Now integrate both sides to obtain

$$x^3y = -\cos x + C,$$

and when $x = \pi$, $y = 0$, thus

$$\begin{aligned} &\pi^3 \cdot 0 = -\cos\pi + C \\ \Rightarrow &0 = 1 + C \\ \Rightarrow &C = -1, \end{aligned}$$

and so

$$\begin{aligned} &x^3y = -\cos x - 1 \\ \Rightarrow &y = -\left(\frac{1+\cos x}{x^3}\right) \end{aligned}$$

is the general solution.

(c) Believe it or not, this ODE is separable! Note that

$$e^{2x-y^2} = e^{2x}e^{-y^2},$$

which will help you to separate the variables. This yields

$$\int ye^{y^2}\,\mathrm{d}y = \int e^{2x}\,\mathrm{d}x.$$

Evaluation of the two integrals results in

$$\frac{1}{2}e^{y^2} = \frac{1}{2}e^{2x} + C$$

Now apply the initial condition $y(0) = 0$ into the above equation. This gives

$$\frac{1}{2} = \frac{1}{2} + C \quad \Rightarrow \quad C = 0,$$

and so

$$\begin{aligned} &\frac{1}{\cancel{2}}e^{y^2} = \frac{1}{\cancel{2}}e^{2x} \\ \Rightarrow &e^{y^2} = e^{2x} \\ \Rightarrow &y^2 = 2x \;\Rightarrow \qquad\qquad y = \sqrt{2x} \end{aligned}$$

is the solution.

(d) The ODE looks almost like the first-order linear form. You can get it into the right form, and find the integrating factor, which is $I(x) = (1 + x)$. But there is a shortcut: you can inspect the original ODE, i.e.

$$(1 + x)\frac{dy}{dx} + y = \sqrt{x},$$

and observe that the LHS is already an exact type! To see this, let's write the LHS as follows:

$$\begin{aligned}(1+x)\frac{dy}{dx}+y &= (1+x)\cdot\frac{dy}{dx}+1\cdot y\\ &= (1+x)\cdot\frac{dy}{dx}+\frac{d}{dx}(1+x)\cdot y\\ &= \frac{d}{dx}\left((1+x)y\right).\end{aligned}$$

Therefore our ODE boils down to

$$\frac{d}{dx}\left((1+x)y\right)=\sqrt{x}\ \left(=x^{\frac{1}{2}}\right),$$

and both sides can be integrated to find that

$$(1+x)y=\frac{2}{3}x^{\frac{3}{2}}+C.$$

As $y(0)=1$, we have

$$\begin{aligned}(1+0)\cdot 1 &= \frac{2}{3}\cdot 1^{\frac{3}{2}}+C\\ \Rightarrow 1 &= \frac{2}{3}+C\\ \Rightarrow C &= \frac{1}{3}.\end{aligned}$$

Therefore

$$(1+x)y=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{3},$$

and, after some rearranging, we arrive at the general solution which is

$$y=\frac{2x^{\frac{3}{2}}+1}{3(1+x)}.$$

2. For this question, you need to construct an ODE which you will then solve. To do so, recall that the slope of a curve is $\frac{dy}{dx}$, hence:

$$\frac{dy}{dx} = -\frac{x}{y}, \qquad y(0) = 1.$$

This ODE can be solved by separation of variables as follows:

$$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x,$$

which gives

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C,$$

and rearranges to

$$x^2 + y^2 = 2C.$$

Then apply the initial condition to find C:

$$0^2 + 1^2 = 2C \quad \Rightarrow \quad 2C = 1,$$

thus

$$x^2 + y^2 = 1,$$

which is a circle (with radius 1 and centre at the origin).

3. (a) When $T > \theta$, we have

$$\theta - T > 0.$$

Moreover, since k is positive, we know that $-k < 0$. Putting all this together shows that

$$\frac{d\theta}{dt} = -k(\theta - T) < 0,$$

so when the object is hotter than its surroundings, it cools down.

(b) The only difference compared to part (a) is that

$$T < \theta \quad \Rightarrow \quad \theta - T < 0,$$

so

$$\frac{d\theta}{dt} = -k(\theta - T) > 0,$$

i.e. when an object is cooler than its surroundings, it will warm up.

(c) This ODE happens to be both separable and a first-order linear equation! That means you have a choice of two routes for solving the ODE. Here we will take the route of separating variables. We obtain

$$\begin{aligned} &\int \frac{1}{\theta - T} \mathrm{d}\theta = \int -k\mathrm{d}t \\ \Rightarrow &\ln|\theta - T| = -kt + C \quad [C = \text{constant}] \\ \Rightarrow &\theta - T = Ge^{-kt} \quad [G = e^C] \\ \Rightarrow &\theta = T + Ge^{-kt} \end{aligned} \tag{2}$$

is the general solution.

(d) Let's gather up all the information given to us by the question:

- We start with $\theta = 100C$ (and let $t = 0$ be the starting time). So $\theta(0) = 100$ is our initial condition.
- The room temperature is $T = 16C$.
- $k = 0.03$.

We can substitute our values for k and T into the general solution (2) from part (c). We obtain

$$\theta(t) = 16 + Ge^{-0.03t}.$$

Now use the initial condition $\theta(0) = 100$ to find G:

$$\begin{aligned} 100 &= 16 + Ge^0 \\ \Rightarrow G &= 100 - 16 = 84. \end{aligned}$$

Hence the temperature of my cup of tea at time t is

$$\theta(t) = 16 + 84e^{-0.03t}.$$

Remember, the tea will be finally comfortable to drink when $\theta = 80$. Substituting this in gives

$$100 = 16 + 84e^{-0.03t},$$

and now we make t the subject.

$$\begin{aligned} 64 &= 84e^{-0.03t} \\ \frac{64}{84} &= e^{-0.03t} \\ -0.03t &= \ln\left(\frac{64}{84}\right), \end{aligned}$$

therefore

$$\begin{aligned} t &= -\frac{1}{0.03}\ln\left(\frac{64}{84}\right) \\ &= 9.065547\ldots \\ &\approx 9\ mboxminutes. \end{aligned}$$