Solutions to Problem Sheet 5

1. (a) Let

$$u = x^2 + 2 \quad \Rightarrow \quad \mathrm{d}u = 2x\,\mathrm{d}x.$$

Then

$$\begin{aligned}
\int x^3 \sqrt{x^2+2}\,\mathrm{d}x &= \int \frac{1}{2}(u-2)u^{\frac{1}{2}}\mathrm{d}u \\
&= \frac{1}{2}\int \left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right] + C \\
&= \frac{1}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{1}{5}\left(x^2+2\right)^{\frac{5}{2}} - \frac{2}{3}\left(x^2+2\right)^{\frac{3}{2}} + C.
\end{aligned}$$

(b) Let

$$u = x^3 \quad \Rightarrow \quad \mathrm{d}u = 3x^2\mathrm{d}x,$$

hence

$$\begin{aligned}
\int x^2\, \cos(x^3)\,\mathrm{d}x &= \int \frac{1}{3}\cos u\,\mathrm{d}u \\
&= \frac{1}{3}\sin u + C \\
&= \frac{1}{3}\sin\left(x^3\right) + C.
\end{aligned}$$

(c) Let

$$u = \frac{1}{x} \quad \Rightarrow \quad \mathrm{d}u = -\frac{1}{x^2}\mathrm{d}x,$$

$$\int \frac{1}{x^2} \sin\left(\frac{1}{x}\right) \mathrm{d}x = \int -\sin u \, \mathrm{d}u$$
$$= \cos u + C$$
$$= \cos\left(\frac{1}{x}\right) + C.$$

(d) Let

$$u = 3x - 7 \quad \Rightarrow \quad \mathrm{d}u = 3\,\mathrm{d}x,$$

$$\int \cos(3x - 7)\,\mathrm{d}x = \int \frac{1}{3} \cos u \, \mathrm{d}u$$
$$= \frac{1}{3} \sin u + C$$
$$= \frac{1}{3} \sin(3x - 7) + C.$$

(e) Let

$$u = \sqrt{x} \quad \Rightarrow \quad \mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x = \frac{1}{2u}\mathrm{d}x$$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int \frac{e^u}{\cancel{u}} 2\cancel{u}\,\mathrm{d}u$$
$$= \int 2e^u \mathrm{d}u$$
$$= 2e^u + C$$
$$= e^{\sqrt{x}} + C.$$

(f) As stated in the question, we will use the following substitution...

$$u = e^x \quad \Rightarrow \quad \mathrm{d}u = e^x \mathrm{d}x = u\,\mathrm{d}x.$$

Also, note that

$$e^{(x+e^x)} = e^x e^{(e^x)} = ue^u,$$

therefore

$$\begin{aligned}\int e^{(x+e^x)}\,\mathrm{d}x &= \int \cancel{u}e^u \frac{1}{\cancel{u}}\,\mathrm{d}u \\ &= \int e^u\,\mathrm{d}u \\ &= e^u + C \\ &= e^{(e^x)} + C.\end{aligned}$$

2. (a) Start by completing the square...

$$\begin{aligned}\int \frac{2}{x^2-6x+10}\,\mathrm{d}x &= \int \frac{2}{(x-3)^2+1}\,\mathrm{d}x \\ &= \int \frac{2}{u^2+1}\,\mathrm{d}u,\end{aligned}$$

where

$$u = x+3 \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x.$$

Then the denominator of (u^2+a^2) with $a=1$ suggests we should substitute:

$$u = \tan\theta \quad \Rightarrow \quad \mathrm{d}u = \sec^2\theta\mathrm{d}\theta.$$

This yields

$$\begin{aligned}
2\int \frac{1}{u^2+1}\,\mathrm{d}u &= 2\int \frac{1}{\tan^2\theta+1}\sec^2\theta\mathrm{d}\theta \\
&= 2\int \frac{1}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\mathrm{d}\theta \\
&= 2\int \mathrm{d}\theta \\
&= 2\theta + C \\
&= 2\tan^{-1} u + C \\
&= 2\tan^{-1}(x+3) + C.
\end{aligned}$$

Note: This integral can be tackled with just one substitution if you let

$$\tan\theta = x+3 \quad \Rightarrow \quad \sec^2\theta\mathrm{d}\theta = \mathrm{d}x.$$

(b) Start by completing the square:

$$\begin{aligned}
\int \frac{1}{\sqrt{-x^2+4x-3}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{1-(x-2)^2}}\,\mathrm{d}x \\
&= \int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u,
\end{aligned}$$

with

$$u = x-2 \quad \Rightarrow \quad \mathrm{d}u = \mathrm{d}x.$$

Then the factor of $\sqrt{1-u^2}$ suggests that we should make a second substitution $u = a\sin\theta$ with $a=1$, i.e.

$$u = \sin\theta \quad \Rightarrow \quad \mathrm{d}u = \cos\theta\,\mathrm{d}\theta,$$

and so we can rewrite the integral as

$$\begin{aligned}
\int \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u &= \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\cancel{\cos\theta}}\cancel{\cos\theta}\,\mathrm{d}\theta \\
&= \int \mathrm{d}\theta \\
&= \theta + C \\
&= \sin^{-1} u + C \\
&= \sin^{-1}(x-2) + C.
\end{aligned}$$

(c) Observe that

$$\begin{aligned}
\int \frac{2}{\sqrt{4x^2-9}}\,\mathrm{d}x &= \int \frac{2}{\sqrt{(2x)^2-3^2}}\,\mathrm{d}x \\
&= \int \frac{\cancel{2}}{\cancel{2}\sqrt{x^2-\left(\frac{3}{2}\right)^2}}\,\mathrm{d}x \\
&= \int \frac{1}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}\,\mathrm{d}x,
\end{aligned}$$

which contains a factor of $\sqrt{x^2-a^2}$ where $a = 3/2$. So substitute

$$x = \frac{3}{2}\cosh\theta \quad \Rightarrow \quad \mathrm{d}x = \frac{3}{2}\sinh\theta\,\mathrm{d}\theta$$

into the integral to obtain

$$\begin{aligned}
\int \frac{2}{\sqrt{4x^2-9}}\,\mathrm{d}x &= \int \frac{2}{\sqrt{(2x)^2-3^2}}\frac{3}{2}\sinh\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sqrt{\left(\frac{3}{2}\cosh\theta\right)^2-\left(\frac{3}{2}\right)^2}}\frac{3}{2}\sinh\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\cancel{\frac{3}{2}}\sqrt{\cosh^2\theta-1}}\cancel{\frac{3}{2}}\sinh\theta\,\mathrm{d}\theta \\
&= \int \frac{\sinh\theta}{\sqrt{\cosh^2\theta-1}}\mathrm{d}\theta,
\end{aligned}$$

and since $\cosh^2\theta - \sinh^2\theta \equiv 1$,

$$\begin{aligned}
\int \frac{2}{\sqrt{4x^2-9}}\,\mathrm{d}x &= \int \frac{\cancel{\sinh\theta}}{\cancel{\sinh\theta}}\mathrm{d}\theta \\
&= \int \mathrm{d}\theta \\
&= \theta + C \\
&= \cosh^{-1}\left(\frac{2x}{3}\right) + C.
\end{aligned}$$

3. (a) If one has $f(x) = \cos x$, then we can make the

following observation...

$$\begin{aligned}\int \tan x \, \mathrm{d}x &= \int -\frac{-\sin x}{\cos x} \mathrm{d}x \\ &= -\int \frac{f'(x)}{f(x)} \mathrm{d}x \\ &= \ln |f(x)| + C \\ &= \ln |\cos x| + C\end{aligned}$$

(b) Using

$$\sin^2 x + \cos^2 x \equiv 1 \quad \Rightarrow \quad \sin^2 x \equiv 1 - \cos^2 x,$$

we have

$$\begin{aligned}\int \sin^5 x \, \mathrm{d}x &= \int \sin^4 x \sin x \, \mathrm{d}x \\ &= \int \left(1 - \cos^2 x\right)^2 \sin x \mathrm{d}x.\end{aligned}$$

Looking at the rewritten integral, we would like to have a substitution that gives

$$\mathrm{d}u = \pm \sin x \mathrm{d}x,$$

which suggests we should choose

$$u = \cos x \quad \Rightarrow \quad \mathrm{d}u = -\sin x \mathrm{d}x.$$

Then

$$\begin{aligned}\int \sin^5 x \,\mathrm{d}x &= -\int \left(1-u^2\right)^2 \mathrm{d}u \\ &= \int \left(-u^2+2u^2-1\right) \mathrm{d}u \\ &= -\frac{1}{5}u^5+\frac{2}{3}u^3-u+C \\ &= -\frac{1}{5}\cos^5 x+\frac{2}{3}\cos^3 x-\cos x+C.\end{aligned}$$

(c) Note that

$$\begin{aligned}\int \frac{1}{x\ln x}\,\mathrm{d}x &= \int \frac{\left(\frac{1}{x}\right)}{\ln x}\,\mathrm{d}x \\ &= \int \frac{f'(x)}{f(x)}\,\mathrm{d}x,\end{aligned}$$

for $f(x)=\ln x$. Hence

$$\begin{aligned}\int \frac{1}{x\ln x}\,\mathrm{d}x &= \ln|f(x)|+C \\ &= \ln|\ln x|+C\end{aligned}$$

4. To find the expected money made, evaluate

$$\begin{aligned}\int_0^3 \frac{dr}{dx}\mathrm{d}x &= \int_0^3 \left[2-\frac{2}{(x+1)^2}\right]\mathrm{d}x \\ &= \left[2x+\frac{2}{x+1}\right]_0^3 \\ &= \left(2\cdot 3+\frac{2}{3+1}\right)-(2\cdot 0+\frac{2}{0+1}) \\ &= 6.5-2 \\ &= 4.5,\end{aligned}$$

i.e. £4500 for 3000 whisks.