Solutions to Problem Sheet 4

1. (a) Start by using the Chain Rule. Then

$$\frac{\partial f}{\partial x} = \frac{2x + 0}{x^2 + y} = \frac{2x}{x^2 + y},$$

where y has been treated as a constant.
In a similar fashion,

$$\frac{\partial f}{\partial y} = \frac{0 + 1}{x^2 + y} = \frac{1}{x^2 + y}.$$

(b) Both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ can be computed by Quotient Rule using $u = x + y$, $v = xy - 1$. Then

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(\cancel{xy} - 1) \cdot 1 - (\cancel{x} + y) \cdot y}{(xy - 1)^2} \\ &= \frac{-(1 + y^2)}{(xy - 1)^2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{(\cancel{xy} - 1) \cdot 1 - (x + \cancel{y}) \cdot x}{(xy - 1)^2} \\ &= \frac{-(1 + x^2)}{(xy - 1)^2}.\end{aligned}$$

(c) Using logarithmic differentiation for $f(x, y) = y^x$,

$$\ln f = x \ln y, \tag{1}$$

hence if we differentiate both sides w.r.t. x,

$$\begin{aligned}\frac{1}{f}\frac{\partial f}{\partial x} &= 1 \cdot \ln y \\ \frac{\partial f}{\partial x} &= f \ln y \\ \frac{\partial f}{\partial x} &= y^x \ln y.\end{aligned}$$

Meanwhile, if we differentiate both sides of (1) w.r.t y instead, we have

$$\begin{aligned}\frac{1}{f}\frac{\partial f}{\partial y} &= x \cdot \frac{1}{y} \\ \frac{\partial f}{\partial y} &= \frac{xf}{y} \\ \frac{\partial f}{\partial y} &= xy^{(x-1)}.\end{aligned}$$

(d) Using the Chain Rule with

$$f(u) = \sinh(u), \quad u = y - 2x,$$

we get

$$\frac{\partial f}{\partial x} = f'(u)\frac{\partial u}{\partial x} = \cosh u \cdot (-2) = -2\cosh(y - 2x)$$

and

$$\frac{\partial f}{\partial y} = f'(u)\frac{\partial u}{\partial y} = \cosh u \cdot 1 = \cosh(y - 2x).$$

(e) This function can be handled using the Product and Chain rules...

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{xy}\right)\ln y + e^{xy}\frac{\partial}{\partial x}\left(\ln y\right) \tag{2}$$
$$= \left(ye^{xy}\right)\ln y + 0 \tag{3}$$
$$= ye^{xy}\ln y. \tag{4}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(e^{xy}\right)\ln y + e^{xy}\frac{\partial}{\partial y}\left(\ln y\right)\\ &= \left(xe^{xy}\right)\ln y + e^{xy}\frac{1}{y}\\ &= e^{xy}\left(x\ln y + \frac{1}{y}\right)\\ &= \frac{e^{xy}}{y}\left(xy\ln y + 1\right).\end{aligned}$$

(f) Using the Chain Rule with

$$f(u) = u^3, \quad u = 2x - 3y,$$

we get

$$\frac{\partial f}{\partial x} = f'(u)\frac{\partial u}{\partial x} = 3u^2 \cdot 2 = 6(2x-3y)^2$$

and

$$\frac{\partial f}{\partial y} = f'(u)\frac{\partial u}{\partial y} = 3u^2 \cdot (-3) = -9(2x-3y)^2.$$

2. Put $u = x - y$. Then

$$\frac{\partial g}{\partial x} = 1 \cdot f(u) + xf'(u)\frac{\partial u}{\partial x} \tag{5}$$
$$= f(x-y) + xf'(x-y) \cdot 1 \tag{6}$$
$$= f(x-y) + xf'(x-y), \tag{7}$$

or

$$(f + xf')$$

for short. Similarly,

$$\frac{\partial g}{\partial y} = 0 \cdot f(u) + xf'(u)\frac{\partial u}{\partial y} \tag{8}$$
$$= 0 + xf'(x-y) \cdot (-1) \tag{9}$$
$$= -xf'(x-y) \tag{10}$$
$$= -xf'. \tag{11}$$

Adding our two results together, we obtain

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = f = \frac{1}{x}g(x,y),$$

i.e.

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = f = \frac{1}{x}g(x,y),$$

hence we have our desired result

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = \frac{1}{x}g.$$

3. First, we need to evaluate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$...

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x) \cdot e^{xy} + x \cdot \frac{\partial}{\partial x}\left(e^{xy}\right) \\
&= 1 \cdot e^{xy} + x \cdot ye^{xy} \\
&= e^{xy}(1 + xy) \\
\frac{\partial f}{\partial y} &= x \cdot \frac{\partial}{\partial y}\left(e^{xy}\right) \\
&= x \cdot xe^{xy} \\
&= x^2 e^{xy}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial y}\left(e^{xy}(1 + xy)\right) \\
&= xe^{xy} \cdot (1 + xy) + e^{xy} \cdot x \\
&= e^{xy}(2x + x^2 y) \\
&= xe^{xy}(2 + xy).
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial x}{\partial\left(\frac{\partial f}{\partial y}\right)} \\
&= \frac{\partial}{\partial x}\left(x^2 e^{xy}\right) \\
&= 2x \cdot e^{xy} + x^2 \cdot ye^{xy} \\
&= xe^{xy}(2 + xy) \\
&= \frac{\partial^2 f}{\partial y \partial x},
\end{aligned}$$

as promised!

4. (a) For $u(x, y) = \ln(x^2 + y^2)$, the two partial derivatives are:

$$u_x = \frac{2x}{x^2 + y^2}, \quad u_y = \frac{2y}{x^2 + y^2}.$$

Next, use the Quotient Rule to find u_{xx}, which gives...

$$\begin{aligned} u_{xx} &= \frac{(x^2 + y^2)(2) - (2x)(2x)}{(x^2 + y^2)^2} \\ &= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} \\ &= \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}. \end{aligned}$$

Similarly, you also end up with

$$u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Therefore

$$u_{xx} + u_{yy} = \frac{\cancel{2(y^2 - x^2)} + \cancel{2(x^2 - y^2)}}{(x^2 + y^2)^2} = 0,$$

i.e. the flow <u>is</u> incompressible and irrotational.

(b) For this flow,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left((x^2 + y^2)^{-\frac{1}{2}}\right) \\ &= -\frac{1}{\cancel{2}}(x^2 + y^2)^{-\frac{3}{2}}(\cancel{2}x) \\ &= \frac{-x}{(x^2 + y^2)^{\frac{3}{2}}}, \end{aligned}$$

and so, differentiating again w.r.t. x,

$$
\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2)^{\frac{3}{2}}}\right) \\
&= \frac{(x^2+y^2)^{\frac{3}{2}}(-1) - (-x)\frac{3}{\cancel{2}}(x^2+y^2)^{\frac{1}{2}}(\cancel{2}x)}{(x^2+y^2)^3} \\
&= \frac{-(x^2+y^2)^{+}3x^2}{(x^2+y^2)^{\frac{5}{2}}} \\
&= \frac{2x^2-y^2}{(x^2+y^2)^{\frac{5}{2}}}
\end{aligned}
$$

Similarly, it can be proved that

$$u_{yy} = \frac{2y^2-x^2}{(x^2+y^2)^{\frac{5}{2}}},$$

and so

$$
\begin{aligned}
u_{xx} + u_{yy} &= \frac{(2x^2-y^2)+(2y^2-x^2)}{(x^2+y^2)^{\frac{5}{2}}} \\
&= \frac{x^2+y^2}{(x^2+y^2)^{\frac{5}{2}}} \\
&= \frac{1}{(x^2+y^2)^{\frac{3}{2}}} \neq 0.
\end{aligned}
$$

thus the flow is <u>not</u> incompressible and irrotational.

(c) We can tell that

$$
\begin{aligned}
u_x &= 2y - 0 + 0 = 2y, \\
u_y &= 2x - 1 + 0 = 2x - 1,
\end{aligned}
$$

and hence

$$u_{xx} = 0, \quad u_{yy} = 0 \quad \Rightarrow \quad u_{xx} + u_{yy} = 0,$$

so the flow is incompressible and irrotational.

(d) For this flow,

$$\begin{aligned} u_x &= -e^{-2y}\sin x, \\ u_y &= -2e^{-2y}\cos x, \end{aligned}$$

and

$$\begin{aligned} u_{xx} &= -e^{-2y}\cos x, \\ u_{yy} &= -4e^{-2y}\cos x, \end{aligned}$$

hence

$$u_{xx} + u_{yy} = -5e^{-2y}\cos x \neq 0,$$

which means the flow is not incompressible and irrotational.