Solutions to Problem Sheet 3

1. Squaring the two equations

$$\begin{aligned} \cosh x + \sinh x &\equiv e^{x} \\ \cosh x - \sinh x &\equiv e^{-x} \end{aligned}$$

yields

$$\cosh^2 x + 2\sinh x \cosh x + \sinh^2 x \equiv e^{2x} \tag{1}$$
$$\cosh^2 x - 2\sinh x \cosh x + \sinh^2 x \equiv e^{2x}. \tag{2}$$

Then we can add Equations (1) and (2) to obtain

$$2\cosh^2 x + 2\sinh^2 x \equiv e^{2x} + e^{-2x}.$$

Finally, divide both sides by 2 to give

$$\cosh^2 x + \sinh^2 x \equiv \cosh 2x,$$

which is the desired identity.

2. (a) We will need the definition

$$\sinh x = \frac{e^x - e^{-x}}{2},$$

which tells us that the LHS is

$$\sinh 3x = \frac{e^{3x} - e^{-3x}}{2} = \frac{(e^x)^3 - (e^{-x})^3}{2}$$

and

$$\begin{aligned} \sinh^3 x &= \left(\frac{e^x - e^{-x}}{2}\right)^3 \\ &= \frac{(e^x)^3 - 3e^x + 3e^{-x} - (e^{-x})^3}{8}. \end{aligned}$$

Then

$$\begin{aligned} 4\sinh^3 x &= 4\left(\frac{(e^x)^3 - 3e^x + 3e^{-x} - (e^{-x})^3}{8}\right) \\ &= \frac{(e^x)^3 - 3e^x + 3e^{-x} - (e^{-x})^3}{2} \\ &= \frac{(e^x)^3 - (e^{-x})^3}{2} - 3\,\frac{e^x - e^{-x}}{2}, \end{aligned}$$

and the RHS is

$$3\sinh x + 4\sinh^3 x = \frac{(e^x)^3 - (e^{-x})^3}{2},$$

therefore LHS≡RHS, and so

$$\sinh 3x \equiv 3\sinh x + 4\sinh^3 x.$$

Remark: This is analogous with the trigonometric identity

$$\sin 3x \equiv 3\sin x - 4\sin^3 x,$$

except that the sign in the RHS has been flipped. This is predicted by Osborne's rule, since the term $4\sin^3 x$ contains a product of two sines.

(b) Again, start with the definition

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

then consider the LHS, i.e.

$$\cosh 3x = \left(\frac{e^{3x} + e^{-3x}}{2}\right) = \frac{(e^x)^3 + (e^{-x})^3}{2}.$$

Meanwhile

$$\begin{aligned}\cosh^3 x &= \left(\frac{e^x+e^{-x}}{2}\right)^3 \\ &= \frac{(e^x)^3+3e^x+3e^{-x}+(e^{-x})^3}{8},\end{aligned}$$

thus

$$\begin{aligned}4\cosh^3 x &= 4\left(\frac{(e^x)^3+3e^x+3e^{-x}+(e^{-x})^3}{8}\right) \\ &= \frac{(e^x)^3+3e^x+3e^{-x}+(e^{-x})^3}{2} \\ &= \frac{(e^x)^3+(e^{-x})^3}{2}+3\,\frac{e^x+e^{-x}}{2},\end{aligned}$$

and the RHS is

$$4\cosh^3 x-3\cosh x=\frac{(e^x)^3+(e^{-x})^3}{2},$$

so LHS≡RHS. Thus

$$\cosh 3x \equiv 4\cosh^3 x - 3\cosh x.$$

Remark: The analogous trigonometric identity is

$$\cos 3x \equiv 4\cos^3 x - 3\cos x.$$

Note that since there are no products of sines, all of the signs match up perfectly, as predicted by Osborne's Rule.

3. (a) We begin by rewriting $y = \coth^{-1} x$ as

$$x = \coth y = \frac{e^y + e^{-y}}{e^y - e^{-y}}.$$

Next,

$$\begin{aligned} & x(e^y - e^{-y}) &= (e^y + e^{-y}) \\ \Rightarrow\quad & x(e^{2y} - 1) &= (e^{2y} + 1) \\ \Rightarrow\quad & e^{2y}(x - 1) &= x + 1 \\ \Rightarrow\quad & (e^y)^2 (x - 1) &= x + 1, \end{aligned}$$

which rearranges to give

$$(e^y)^2 (x - 1) - (x + 1) = 0,$$

which is a quadratic equation for e^y (with $a = (x-1)$, $b = 0$, $c = -(x+1)$. Solving this for e^y gives

$$e^y = \pm\sqrt{\frac{x+1}{x-1}},$$

but $e^y > 0$, so we choose the positive square root and

$$e^y = \sqrt{\frac{x+1}{x-1}} \qquad \Rightarrow \qquad y = \ln\left(\sqrt{\frac{x+1}{x-1}}\right),$$

i.e.

$$y = \ln\left[\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}}\right] \qquad \Rightarrow \qquad y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right),$$

which is the desired result. Note that any other valid method which gives this result will also be accepted.

(b) Using Equation (1) from part (a), we see that for $x = \frac{5}{4}$,

$$\begin{aligned} y &= \coth^{-1}\left(\frac{5}{4}\right) \\ &= \frac{1}{2}\ln\left(\frac{\frac{5}{4}+1}{\frac{5}{4}-1}\right) \\ &= \frac{1}{2}\ln\left(\frac{9/\cancel{4}}{1/\cancel{4}}\right) \\ &= \frac{1}{2}\ln 9 \\ &= \ln 3. \end{aligned}$$

4. Here is a sketch of the scenario...

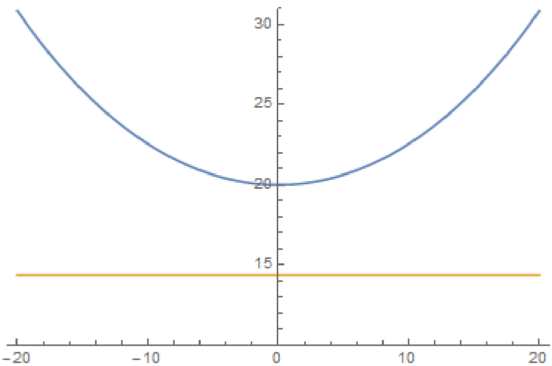

Figure 1: A plot of the shape of the live cable; this is given by the blue curve. Also, if you thought the pavement (shown in orange) was on the x-axis, think again! The y-coordinate of the pavement is $20\cosh(1) - 16.5 \approx 14.3$ feet, not zero.

If $a = 20$ then the equation becomes

$$y(x) = 20\cosh\left(\frac{x}{20}\right).$$

The poles are at $x = \pm 20$, where

$$y(\pm 20) = 20\cosh(1) \approx 30.8$$

(remembering that $\cosh(x)$ is an even function, so $\cosh(-x) = \cosh(x)$.)

The lowest point of the curve is at the minimum, where $\frac{dy}{dx} = 0$:

$$\begin{aligned} 0 &= y'(x) \\ &= \frac{20}{20}\sinh\left(\frac{x}{20}\right), \end{aligned}$$

i.e. when $x = 0$. This is clear anyway from thinking about the shape of the cosh graph.
When $x = 0$,

$$y(0) = 20\cosh(0) = 20.$$

So the difference in height between the top of the cable and the bottom of the cable is $30.8 - 20 = 10.8$ feet $= 10'\,9\frac{1}{2}''$.

The cable hangs from the tops of poles that are $16'\,6''$ tall, so the lowest point of the cable is $16'\,6'' - 10'\,9\frac{1}{2}'' = 5'\,8\frac{1}{2}''$ off the ground. So Carl, at $6'\,1\frac{1}{2}''$, will definitely get fried, while Amanda, at $5'\,3\frac{1}{2}''$, will survive (but she will probably be shocked emotionally, even if not electrically!)