Handout: Integration

Basic integrals

$f(x)$	$\int f(x)\,dx$
$x^n \ (n \neq -1)$	$\frac{1}{n+1}x^{n+1} + C$
x^{-1}	$\ln\lvert x\rvert + C$
e^{ax}	$\frac{1}{a}e^{ax} + C$
$\cos(ax)$	$\frac{1}{a}\sin(ax) + C$
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + C$

Integration by change of variables

Sometimes changing our integration variable can make our integral much easier to do. Typically, we change from x to u, using the procedure below.

$$\int f(x)\,dx = \int f(x)\frac{dx}{du}\,du \qquad (\star)$$

1. Choose a new variable $u = g(x)$ and write $f(x)$ in terms of u.
2. Calculate $\dfrac{dx}{du} = \dfrac{1}{g'(u)}$ and write it in terms of u.
3. You now should have the right hand side of $(\star)$, all in terms of u.
4. Compute the u integral.
5. Rewrite in terms of x.

Standard results derived using change of variables

If you see	Try substituting
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\sinh\theta$
$\sqrt{x^2 - a^2}$	$x = a\cosh\theta$
$\dfrac{1}{a^2 + x^2}$	$x = a\tan\theta$

Integration by parts

We use integration by parts to integrate a product of functions that are **not of the above form!** The formula is

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx} \tag{1}$$

Generally, if one of the functions is a polynomial, we take it as $u(x)$. We can also use integration by parts to integrate inverse functions, including $\ln x$ and inverse trigonometric and hyperbolic functions. The trick is to set $\frac{dv}{dx} = 1$.

Integrals involving roots of quadratics in the denominator

If we have an integral involving the square root of a quadratic in the denominator, then the method will depend on what is in the numerator! If the numerator is a constant, then you can use one of the standard results in the table. You may have to complete the square and do an appropriate substitution first.

Integrating Rational Functions

First, check if you can factorise the denominator. If yes, then decompose into partial fractions! If no, then complete the square to get it into the following form.

$$ax^2 + bx + c = a((x+\alpha)^2 + \beta^2) \tag{2}$$

Then change variables to u by using the following substitution

$$x + \alpha = \beta u \tag{3}$$

We then have a quadratic in the standard form $\pm(u^2+1)$. After this, we may have multiple fractions. Again, to integrate these, we may have to use different methods! If we have a constant in the numerator, we then use the result for $\tan^{-1}$.

However if the numerator is not constant, but looks like $C + Du$, we rewrite our fraction as a sum of two parts and handle each one separately. The first part should have a constant for the numerator, namely C. Meanwhile, the other part will have Du for the numerator, whose integral should give a log (i.e. $\ln(u^2+1)$) in return.