# MATH6501 Mathematics for Engineers 1 

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## Chapter 1

## Differentiation

### 1.1 Introduction

Why differentiation? Well, it is a useful tool because many real-world problems rely on the rates of change of quantities. For example, speed is the rate of change of distance of a moving object.

Sometimes an engineer will need to look at a graph of, for example, distance vs time. In that case, questions about rate of change become questions about gradients, i.e. slopes of the tangent to a curve.


Slope of the chord PQ

$$
=\frac{\text { Change in } y}{\text { Change in } x}=\frac{f(x+\delta x)-f(x)}{\delta x},
$$

and as $\delta x \rightarrow 0$, chord $\rightarrow$ tangent.
Therefore: Slope of the tangent at $x$

$$
=\frac{\mathrm{d} y}{\mathrm{~d} x}=\lim _{\delta x \rightarrow 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)
$$

Example 1.1. Use the above definition to differentiate $y=f(x)=x^{2}$.

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\lim _{\delta x \rightarrow 0}\left(\frac{(x+\delta x)^{2}-x^{2}}{\delta x}\right) \\
& =\lim _{\delta x \rightarrow 0}\left(\frac{\not x^{\mathscr{}}+2 x \delta x+(\delta x)^{2}-\not x^{\mathscr{}}}{\delta x}\right) \\
& =\lim _{\delta x \rightarrow 0}(2 x+\delta x) \\
& =2 x .
\end{aligned}
$$

### 1.2 Basic differentiation

Now let's consider the functions given in Table 1.1. These are the basic building blocks of the many functions an engineer will need to differentiate (chances are you already saw these in A-Level).

Let us start by calculating some basic derivatives...
Example 1.2. Compute

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(2 e^{x}-3 \cos x\right)
$$

Applying the addition formula (Rule 1 in Table 1.2) yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left(2 e^{x}-3 \cos x\right) & =2 \frac{\mathrm{~d}}{\mathrm{~d} x}\left(e^{x}\right)-3 \frac{\mathrm{~d}}{\mathrm{~d} x}(\cos x) \\
& =2 e^{x}-3(-\sin x) \\
& =2 e^{x}+3 \sin x
\end{aligned}
$$

So we can find derivatives for sums of functions. However, if we are handling a product of functions, we need the Product Rule instead:

| $f(x)$ | $\frac{\mathrm{d} f}{\mathrm{~d} x}$ |
| :---: | :---: |
| $x^{n}$ | $n x^{n-1}$ |
| 1 | 0 |
| $\ln (x)$ | $x^{-1}$ |
| $e^{x}$ | $e^{x}$ |
| $\sin (x)$ | $\cos (x)$ |
| $\cos (x)$ | $-\sin (x)$ |
| $\sinh (x)$ | $\cosh (x)$ |
| $\cosh (x)$ | $\sinh (x)$ |

Table 1.1: Table of Basic Derivatives

| Rule | $f(x)$ | $\frac{\mathrm{d} f}{\mathrm{~d} x}$ | Notes |
| :---: | :---: | :---: | :---: |
| 1 | $u+v$ | $\frac{\mathrm{~d} u}{\mathrm{~d} x}+\frac{\mathrm{d} v}{\mathrm{~d} x}$ | Addition Rule |
| 2 | $C u$ | $C \frac{\mathrm{~d} u}{\mathrm{~d} x}$ | $(C=$ constant $)$ |
| 3 | $u v$ | $v \frac{\mathrm{~d} u}{\mathrm{~d} x}+u \frac{\mathrm{~d} v}{\mathrm{~d} x}$ | Product Rule |
| 4 | $u / v$ | $\frac{v \frac{d u}{\mathrm{~d} x}-u \frac{\mathrm{~d} v}{\mathrm{~d} x}}{v^{2}}$ | Quotient Rule |
| 5 | $f(u(x))$ | $f^{\prime}(u(x)) \frac{\mathrm{d} u}{\mathrm{~d} x}$ | Chain Rule |
| 6 | $\frac{\mathrm{~d} x}{\mathrm{~d} y}$ | $\frac{\frac{1}{\mathrm{~d} y}}{\mathrm{~d} x}$ | For Inverse Functions |

Table 1.2: Table of Rules for Differentiation

Example 1.3. Compute

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{3} \sin x\right)
$$

This is a product of two functions, hence the Product Rule is required (Rule 3 in Table 2). This is:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(u v)=v \frac{\mathrm{~d} u}{\mathrm{~d} x}+u \frac{\mathrm{~d} v}{\mathrm{~d} x} .
$$

For this example, let $u=x^{3}$ and $v=\sin x$. Then we have...

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{3} \sin x\right)=\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{3}\right) \sin x+x^{3} \frac{\mathrm{~d}}{\mathrm{~d} x}(\sin x)
$$

i.e.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{3} \sin x\right)=3 x^{2} \sin x+x^{3} \cos x
$$

The Product Rule still works if you want to compute the derivative of a function that is a product of three or more functions.

Example 1.4. Compute

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{2} e^{x} \sin x\right) \\
= & \frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{2}\right) e^{x} \sin x \\
+ & x^{2} \frac{\mathrm{~d}}{\mathrm{~d} x}\left(e^{x}\right) \sin x \\
+ & x^{2} e^{x} \frac{\mathrm{~d}}{\mathrm{~d} x}(\sin x) \\
= & \left(2 x e^{x}+x^{2} e^{x}\right) \sin x+x^{2} e^{x} \cos x
\end{aligned}
$$

This next example shows a standard use of the Quotient Rule:
Example 1.5. Compute

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{x-1}{x^{2}+1}\right) .
$$

Applying the Quotient Rule gives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{x-1}{x^{2}+1}\right) & =\frac{\left(x^{2}+1\right) \frac{\mathrm{d}}{\mathrm{~d} x}(x-1)-(x-1) \frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{2}+1\right)}{\left(x^{2}+1\right)^{2}} \\
& =\frac{\left(x^{2}+1\right) \times 1-(x-1) \times 2 x}{\left(x^{2}+1\right)^{2}} \\
& =\frac{-x^{2}+2 x+1}{\left(x^{2}+1\right)^{2}}
\end{aligned}
$$

Example 1.6 (Differentiate $\tanh x$ using the quotient rule).

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}(\tanh x) & =\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{\sinh x}{\cosh x}\right) \\
& =\frac{\cosh x \frac{\mathrm{~d}}{\mathrm{~d} x}(\sinh x)-\sinh x \frac{\mathrm{~d}}{\mathrm{~d} x}(\cosh x)}{\cosh ^{2} x} \\
& =\frac{\cosh \times \cosh x-\sinh x \times \sinh x}{\cosh ^{2} x} \\
& =\frac{\cosh ^{2} x-\sinh ^{2} x}{\cosh ^{2} x},
\end{aligned}
$$

and now using the hyperbolic identity

$$
\cosh ^{2} x-\sinh ^{2} x \equiv 1,
$$

this leads to

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\tanh x)=\frac{1}{\cosh ^{2} x},
$$

and since

$$
\operatorname{sech} x \equiv \frac{1}{\cosh x} \quad \Longrightarrow \quad \operatorname{sech}^{2} x \equiv \frac{1}{\cosh ^{2} x},
$$

this leads to the result

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\tanh x)=\operatorname{sech}^{2} x
$$

This looks very similar to the following result...

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\tan x)=\sec ^{2} x
$$

which uses the trigonometric functions instead of hyperbolic ones. You will get to prove this result for yourself in the Problem Sheet!

### 1.3 The Chain Rule

So far, we have calculated derivatives of sums, products and quotients of functions. But what happens when you have a function of a function?

Example 1.7. Compute the following derivative

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\sin 2 x)
$$

The Chain Rule says that

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(f(u(x)))=f^{\prime}(u(x)) \frac{\mathrm{d} u}{\mathrm{~d} x}
$$

So we let

$$
\begin{array}{ll}
u(x)=2 x, & \frac{\mathrm{~d} u}{\mathrm{~d} x}=2 \\
f(u)=\sin u & \frac{\mathrm{~d} f}{\mathrm{~d} u}=\cos u
\end{array}
$$

then applying the chain rule gives

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\sin 2 x)=\frac{\mathrm{d}}{\mathrm{~d} u}(f(u)) \frac{\mathrm{d} u}{\mathrm{~d} x}=2 \cos u
$$

and rewriting back in terms of the original variable $x$ gives

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\sin 2 x)=2 \cos 2 x
$$

Let's try another example...
Example 1.8. Compute the following derivative

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\ln \left(x^{2}-1\right)\right)
$$

Put

$$
\begin{aligned}
& u(x)=x^{2}-1, \quad u^{\prime}(x)=2 x \\
& f(u)=\ln u, \quad f^{\prime}(u)=\frac{1}{u}
\end{aligned}
$$

then applying the chain rule gives

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(\ln \left(x^{2}-1\right)\right)=\frac{2 x}{u}=\frac{2 x}{x^{2}-1}
$$

You will want to brace yourself for the next example! This one shows you how to use the chain rule more than once.

Example 1.9. Compute the following derivative
$\frac{\mathrm{d}}{\mathrm{d} x}\left(\sin \left(\ln \left(x^{2} e^{x}\right)\right)\right)$
First apply chain rule with $f(u)=\sin u, u=\ln \left(x^{2} e^{x}\right)$
$=\cos \left(\ln \left(x^{2} e^{x}\right)\right) \times \frac{\mathrm{d}}{\mathrm{d} x}\left(\ln \left(x^{2} e^{x}\right)\right)$
Then apply chain rule again, this time with $f(u)=\ln u, u=x^{2} e^{x}$
$=\cos \left(\ln \left(x^{2} e^{x}\right)\right) \frac{1}{x^{2} e^{x}} \frac{\mathrm{~d}}{\mathrm{~d} x}\left(x^{2} e^{x}\right)$
Finally, apply the product rule with $u=x^{2}, v=e^{x}$
$=\cos \left(\ln \left(x^{2} e^{x}\right)\right) \frac{1}{x^{2} e^{x}}\left[x^{2} e^{x}+2 x e^{x}\right]$.

Example 1.10 (2009 Exam Question). Compute the following derivative:

$$
\frac{\mathrm{d} y}{\mathrm{~d} x} \quad \text { for } \quad y=\sin \left(\frac{e^{-x}}{x}\right)
$$

This problem requires the chain rule with

$$
\begin{aligned}
f(u) & =\sin u, \quad \frac{\mathrm{~d} f}{\mathrm{~d} u}=\cos u \\
u & =\frac{e^{-x}}{x}, \quad \frac{\mathrm{~d} u}{\mathrm{~d} x}=-\frac{e^{-x}}{x}-\frac{e^{-x}}{x^{2}} .
\end{aligned}
$$

Hence

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\cos \left(\frac{e^{-x}}{x}\right)\left(-\frac{e^{-x}}{x}-\frac{e^{-x}}{x^{2}}\right)
$$

### 1.3.1 Implicit differentiation

Sometimes you can't write a function in terms of $x$ only. In that case, if you are differentiating w.r.t. $x$, you use implicit differentiation.

Example 1.11 (Slope of a circle with radius 1). Suppose $x^{2}+y^{2}=1$.

- This is the equation of a circle, centre $O$, radius 1 .
- $y$ is an implicit function of $x$, i.e. not in the form

$$
y=\text { Stuff depending on } x \text { only }
$$

- To find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ we take $\frac{\mathrm{d}}{\mathrm{d} x}$ of all terms:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{2}\right)+\frac{\mathrm{d}}{\mathrm{~d} x}\left(y^{2}\right)=\frac{\mathrm{d}}{\mathrm{~d} x}(1)
$$

i.e

$$
2 x+2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0 \quad \therefore \quad \frac{\mathrm{~d} y}{\mathrm{~d} x}=-\frac{x}{y}
$$

Example 1.12. If the equation of a curve satisfies

$$
x^{2}+3 x y+y^{2}=7
$$

find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ in terms of $x$ and $y$.
Proceed by differentiating each term w.r.t. $x$ :

$$
2 x+3 y+3 x \frac{\mathrm{~d} y}{\mathrm{~d} x}+2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}=0
$$

(Common error: Forgetting to differentiate the 7!)

$$
\text { i.e } \frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{2 x+3 y}{3 x+2 y} \text {. }
$$

## Logarithmic differentiation

Sometimes it is useful to take logs on both sides of an equation before differentiating. By doing this you are setting up an implicit equation, making this an example of implicit differentiation.

Example 1.13. Differentiate the function $y=10^{x}$ with respect to $x$.

$$
y=10^{x}, \quad \therefore \quad \ln y=x \ln 10
$$

and so in differentiating w.r.t $x$

$$
\begin{aligned}
\frac{1}{y} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =\ln 10 \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =10^{x} \ln 10
\end{aligned}
$$

Example 1.14. Find

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(x^{x}\right)
$$

First let $y=x^{x}$, then $\ln y=\ln x^{x}=x \ln x$.

$$
\begin{array}{rlrl} 
& & \frac{\mathrm{d}}{\mathrm{~d} x}(\ln y) & =\frac{\mathrm{d}}{\mathrm{~d} x}(x \ln x) \\
& \Rightarrow & \frac{1}{y} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =\ln x+\frac{\not x}{\not x} \\
\Rightarrow & \frac{\mathrm{~d} y}{\mathrm{~d} x} & =y(1+\ln x) \\
& \therefore & & \frac{\mathrm{d} y}{\mathrm{~d} x}
\end{array}=x^{x}(1+\ln x) .
$$

Example 1.15.

$$
y=\frac{x^{2} \cos x}{\sin 2 x} \quad\left(=\frac{x^{2}}{2 \sin x}\right)
$$

Take logs and differentiate with respect to $x$ to give

$$
\begin{aligned}
\ln y & =\ln x^{2}+\ln \cos x-\ln \sin 2 x \\
\frac{1}{y} \frac{\mathrm{~d} y}{\mathrm{~d} x} & =\frac{2 x}{x^{2}}-\frac{\sin x}{\cos x}-2 \frac{\cos 2 x}{\sin 2 x} \\
\therefore \quad \frac{\mathrm{~d} y}{\mathrm{~d} x} & =y\left(\frac{2}{x}-\tan x-2 \cot 2 x\right) \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{x^{2} \cos x}{\sin 2 x}\left(\frac{2}{x}-\tan x-2 \cot 2 x\right)
\end{aligned}
$$

## Differentiating Inverse functions

Believe it or not, when you differentiate an inverse function, you are using implicit differentiation (again!)

## Example 1.16.

Find $\frac{\mathrm{d} y}{\mathrm{~d} x} \quad$ when $\quad y=\sin ^{-1} x$.

$$
\begin{aligned}
y & =\sin ^{-1} x \\
\sin y & =x \\
\frac{\mathrm{~d}}{\mathrm{~d} x}(\sin y) & =1 \\
\cos y \frac{\mathrm{~d} y}{\mathrm{~d} x} & =1 \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^{2}}}
\end{aligned}
$$

Example 1.17.
Find $\frac{\mathrm{d} y}{\mathrm{~d} x} \quad$ when $\quad y=\cosh ^{-1} x$.

$$
\begin{aligned}
y & =\cosh ^{-1} x \\
x & =\cosh y \\
1 & =\sinh y \frac{\mathrm{~d} y}{\mathrm{~d} x} \quad \quad \text { (Implicit differentiation) } \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{1}{\sinh y} \\
& =\frac{1}{\sqrt{\cosh ^{2} y-1}} \quad\left(\cosh ^{2} y-\sinh ^{2} y \equiv 1\right) \\
& =\frac{1}{\sqrt{x^{2}-1}} .
\end{aligned}
$$

Therefore

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{1}{\sqrt{x^{2}-1}}
$$

### 1.4 Higher derivatives

Having found $\frac{\mathrm{d} y}{\mathrm{~d} x}$, we can differentiate this again, which gives the second derivative $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$. If we then differentiate again, we get $\frac{\mathrm{d}^{3} y}{\mathrm{~d} x^{3}}, \frac{\mathrm{~d}^{4} y}{\mathrm{~d} x^{4}}$, etc. These are collectively known as higher derivatives.

## Example 1.18.

$$
\begin{aligned}
y & =x^{6} \\
\frac{\mathrm{~d} y}{\mathrm{~d} x} & =6 x^{5} \\
\frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}} & =6 \times 5 x^{4}=30 x^{4} \\
\frac{\mathrm{~d}^{3} y}{\mathrm{~d} x^{3}} & =30 \times 4 x^{3}=120 x^{3} \\
\frac{\mathrm{~d}^{4} y}{\mathrm{~d} x^{4}} & =360 x^{2} \\
\frac{\mathrm{~d}^{5} y}{\mathrm{~d} x^{5}} & =720 x \\
\frac{\mathrm{~d}^{6} y}{\mathrm{~d} x^{6}} & =720 \\
\frac{\mathrm{~d}^{7} y}{\mathrm{~d} x^{7}} & =0 \\
\frac{\mathrm{~d}^{8} y}{\mathrm{~d} x^{8}} & =0 .
\end{aligned}
$$

For convenience the following notation is sometimes used for higher derivatives:

$$
\begin{gathered}
\frac{\mathrm{d}^{n} y}{\mathrm{~d} x^{n}}=y^{(n)} \\
\text { and so } \frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=y^{(2)}, \quad \frac{\mathrm{d}^{3} y}{\mathrm{~d} x^{3}}=y^{(3)}, \quad \text { etc. }
\end{gathered}
$$

Example 1.19.

$$
\text { For } \begin{aligned}
y=\sin 2 x, & \text { find } \frac{\mathrm{d} y}{\mathrm{~d} x}, \quad \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}, \quad y^{(3)} . \\
\frac{\mathrm{d} y}{\mathrm{~d} x} & =2 \cos 2 x, \\
\frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}} & =-4 \sin 2 x \\
y^{(3)} & =-8 \cos 2 x .
\end{aligned}
$$

Example 1.20. If $y=e^{2 x}$, what is $\frac{\mathrm{d}^{n} y}{\mathrm{~d} x^{n}}$ ?

$$
\begin{gathered}
\frac{\mathrm{d} y}{\mathrm{~d} x}=y^{(1)}=2 e^{2 x}, \quad y^{(2)}=4 e^{2 x}, \quad y^{(3)}=8 e^{2 x} \\
\therefore \quad y^{(n)}=2^{n} e^{2 x} .
\end{gathered}
$$

### 1.4.1 Computing the $n^{\text {th }}$ derivative of a product

Suppose we have a function defined as a product, i.e. given by

$$
y=u v, \quad \text { where } \quad u=u(x), v=v(x)
$$

In general if $y=u v$ then applying the product rule gives:

$$
\begin{aligned}
y^{(1)} & =u^{(1)} v+u v^{(1)} \\
y^{(2)} & =u^{(2)} v+u^{(1)} v^{(1)}+u^{(1)} v^{(1)}+u v^{(2)} \\
y^{(3)} & =u^{(3)} v+3 u^{(2)} v^{(1)}+2 u^{(2)} v^{(1)}+2 u^{(1)} v^{(2)} \\
& +u^{(1)} v^{(2)}+u v^{(3)} \\
& =u^{(3)}+3 u^{(2)} v^{(1)}+3 u^{(1)} v^{(2)}+u v^{(3)} .
\end{aligned}
$$

Notice that the binomial coefficients are appearing.

In fact...

$$
\begin{align*}
y^{(n)} & =u^{(n)} v+\binom{n}{1} u^{(n-1)} v^{(1)}+\binom{n}{2} u^{(n-2)} v^{(2)}+\cdots \\
& +\binom{n}{n-1} u^{(1)} v^{(n-1)}+u v^{(n)} \\
& =\sum_{k=0}^{n}\binom{n}{k} u^{(n-k)} v^{(k)} \tag{1.1}
\end{align*}
$$

where

$$
\binom{n}{k}=\frac{n!}{(n-k)!k!}
$$

Equation 1.1 is known as the Leibniz rule for differentiating a product $n$ times.

## Example 1.21.

$$
\text { If } y=x e^{x}, \quad \text { what is } \quad \frac{\mathrm{d}^{n} y}{\mathrm{~d} x^{n}} \quad ?
$$

Using the Leibniz rule with $v=x, u=e^{x}$ gives

$$
\begin{aligned}
y^{(n)} & =x \frac{\mathrm{~d}^{n}}{\mathrm{~d} x^{n}}\left(e^{x}\right)+\binom{n}{1} \frac{\mathrm{~d}}{\mathrm{~d} x}(x) \frac{\mathrm{d}^{n-1}}{\mathrm{~d} x^{n-1}}\left(e^{x}\right) \\
& +\binom{n}{2} \frac{\mathrm{~d}^{2}}{\mathrm{~d} x^{2}}(x) \frac{\mathrm{d}^{n-2}}{\mathrm{~d} x^{n-2}}\left(e^{x}\right)+0 \\
& =x e^{x}+n .1 . e^{x} \\
& =e^{x}(x+n)
\end{aligned}
$$

## Example 1.22.

$$
\text { Let } y=x^{2} \sin x . \quad \text { Find } \quad \frac{\mathrm{d}^{17} y}{\mathrm{~d} x^{17}}
$$

Tip: When applying the Leibniz rule for the function $u v$ you should choose $v$ such that it becomes zero when differentiated a relatively few number of times (if this is possible). So we choose $u=\sin x, v=x^{2}$.

$$
\begin{aligned}
y^{(17)} & =x^{2} \frac{\mathrm{~d}^{17}}{\mathrm{~d} x^{17}}(\sin x)+\binom{17}{1} 2 x \frac{\mathrm{~d}^{16}}{\mathrm{~d} x^{16}}(\sin x) \\
& +\binom{17}{2} 2 \frac{\mathrm{~d}^{15}}{\mathrm{~d} x^{15}}(\sin x)+0
\end{aligned}
$$

Now it can be shown that

$$
\begin{aligned}
\frac{\mathrm{d}^{16}}{\mathrm{~d} x^{16}}(\sin x) & =\sin x, \quad \therefore \quad \frac{\mathrm{~d}^{17}}{\mathrm{~d} x^{17}}(\cos x), \quad \frac{\mathrm{d}^{15}}{\mathrm{~d} x^{15}}(-\cos x) . \\
\therefore \quad y^{(17)} & =x^{2} \cos x+17.2 x \sin x+\frac{17.16}{22} \cdot 22 \cdot(-\cos x) \\
& =x^{2} \cos x+34 x \sin x-272 \cos x .
\end{aligned}
$$

### 1.4.2 Parametric differentiation

In many applications a function is expressed using a PARAMETER, e.g.

$$
y=\cos 2 t, \quad x=\sin t
$$

where the parameter $t \equiv$ time (for example).

- For a given value of $t$, both $x$ and $y$ may be found.
- This implies that we can generate a curve $y=f(x)$.

Example 1.23. If a curve is defined parametrically as

$$
y=\cos 2 t, \quad x=\sin t, \quad \text { then find } \quad \frac{\mathrm{d} y}{\mathrm{~d} x} \quad \text { and } \quad \frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}
$$

First,

$$
\begin{aligned}
& \frac{\mathrm{d} y}{\mathrm{~d} t}=-2 \sin 2 t \quad \text { and } \quad \frac{\mathrm{d} x}{\mathrm{~d} t}=\cos t . \\
& \text { Thus } \frac{\mathrm{d} y}{\mathrm{~d} x}=\underbrace{\frac{\mathrm{d} y}{\mathrm{~d} t} \cdot \frac{\mathrm{~d} t}{\mathrm{~d} x}}_{\text {Chain Rule }}=\frac{\frac{\mathrm{d} y}{\mathrm{~d} t}}{\frac{\mathrm{~d} x}{\mathrm{~d} t}} .
\end{aligned}
$$

Then

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{-2 \sin 2 t}{\cos t}=-\frac{4 \sin t \cos t}{\cos t}=-4 \sin t
$$

What about...?

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}} \quad\left(\neq \frac{\mathrm{d}^{2} y}{\mathrm{~d} t^{2}} / \frac{\mathrm{d}^{2} x}{\mathrm{~d} t^{2}}\right)
$$

By definition

$$
\begin{aligned}
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=\frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{\mathrm{~d} y}{\mathrm{~d} x}\right) & =\frac{\mathrm{d}}{\mathrm{~d} x}(-4 \sin t) \\
& =\frac{\mathrm{d}}{\mathrm{~d} t}(-4 \sin t) \frac{\mathrm{d} t}{\mathrm{~d} x} \quad \text { (Chain Rule!) } \\
& =-4 \frac{\cos t}{\frac{\mathrm{~d} x}{\mathrm{~d} t}}=-\frac{4 \cos t}{\cos t}=-4
\end{aligned}
$$

## Example 1.24.

$$
y=3 \sin \theta-\sin ^{3} \theta, \quad x=\cos ^{3} \theta, \quad \text { Find } \quad \frac{\mathrm{d} y}{\mathrm{~d} x}, \quad \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}
$$

In this example $\theta$ is the parameter.

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{\mathrm{d} y}{\mathrm{~d} \theta} / \frac{\mathrm{d} x}{\mathrm{~d} \theta}=\frac{\not 2 \cos \theta-\not 2 \sin ^{2} \theta \cos \theta}{-\not 2 \cos ^{2} \theta \sin \theta} \\
& =\frac{\cos \theta\left(1-\sin ^{2} \theta\right)}{-\cos ^{2} \theta \sin \theta}=\frac{\cos \theta\left(\cos ^{2} \theta\right)}{-\cos ^{2} \theta \sin \theta} \\
& =-\frac{\cos \theta}{\sin \theta}=-\cot \theta
\end{aligned}
$$

Meanwhile,

$$
\begin{aligned}
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}} & =\frac{\mathrm{d}}{\mathrm{~d} x}(-\cot \theta)=\frac{\mathrm{d}}{\mathrm{~d} \theta}(-\cot \theta) \frac{\mathrm{d} \theta}{\mathrm{~d} x} \\
& =-\left(-\frac{1}{\sin ^{2} \theta}\right) /\left(-3 \cos ^{2} \theta \sin \theta\right) \\
& =-\frac{1}{3 \cos ^{2} \theta \sin ^{3} \theta}
\end{aligned}
$$

### 1.5 Using differentiation

### 1.5.1 Finding stationary points

Consider the following diagram...


First observe that

1. If $f^{\prime}(a)<0$ then $f$ is decreasing near $a$,
2. If $f^{\prime}(b)>0$ then $f$ is increasing near $b$.

A stationary point is where $\frac{\mathrm{d} y}{\mathrm{~d} x}=0$. It can correspond to either...

| $\frac{\mathrm{d} y}{\mathrm{~d} x}$ | $\frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}$ | $\frac{\mathrm{~d} y}{\mathrm{~d} x}$ is | Classification |
| :---: | :--- | :--- | :--- |
| 0 | $>0$ | $\Rightarrow$ Increasing | $\Rightarrow$ Minimum |
| 0 | $<0$ | $\Rightarrow$ Decreasing | $\Rightarrow$ Maximum |
| 0 | $=0$ | $\Rightarrow$ ??? | $\Rightarrow$ Need more info! |

Table 1.3: Using second derivatives to classify stationary points

1. A maximum (derivative changes from positive to negative)
2. A minimum (derivative changes from negative to positive)
3. A point of inflection (second derivative changes sign)


Remark 1.1. A point of inflection does not have to be a stationary point. So watch out!

Second Derivative Tests for stationary points...
Example 1.25. For

$$
y=x^{4}, \quad \frac{\mathrm{~d} y}{\mathrm{~d} x}=4 x^{3}
$$

$\therefore \quad$ Stationary point at $x=0$.

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=12 x^{2}=0 \quad \text { at } \quad x=0
$$

But clearly $x=0$ is a minimum, as shown in Figure 1.1.
... hence we need a different test. Fortunately, we do have one. . . we can construct a sign diagram of $\frac{\mathrm{d} y}{\mathrm{~d} x}$, as done in Figure 1.2. This works even when $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=0$ !

Hence the point $x=0$ must be a minimum.
Example 1.26. Find all the stationary points and their nature for

$$
y=f(x)=3 x^{4}-4 x^{3}+1
$$



Figure 1.1: A plot of $y=x^{4}$. We can see that there is a minimum at $x=0$; however, the usual second derivative test doesn't work on this one!


Figure 1.2: The sign test for $y=x^{4}$. This is done by checking the sign of $\frac{\mathrm{d} y}{\mathrm{~d} x}$ on either side of the stationary point $x=0$., which tells you whether the he tangent to the curve points up or down for each side of the stationary point.

Calculating the first derivative yields

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=12 x^{3}-12 x^{2}=12 x^{2}(x-1)
$$

At the stationary points

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=0, \quad \text { and so } \quad 12 x^{2}(x-1)=0
$$

$\therefore \quad$ Stationary points at $x=0,1$.
Now apply the second derivative test. Calculating the second derivative yields

$$
\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}=36 x^{2}-24 x
$$

Evaluating the value of the second derivative at the stationary points gives

$$
\begin{aligned}
& \text { At } \quad x=1 \quad \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}=36-24>0 \quad \therefore \text { Minimum. } \\
& \text { At } \quad x=0 \quad \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}=0 \quad \therefore \quad \text { Use different test. }
\end{aligned}
$$

For the point $x=0$, construct a sign diagram for $\frac{\mathrm{d} y}{\mathrm{~d} x}$, as done in Figure 1.3
Therefore $x=1$ is a minimum, while $x=0$ is a point of inflection.


Figure 1.3: Sign test for the derivative of $3 x^{4}-4 x^{3}+1$, which demonstrates that $x=0$ has a point of inflection.

Example 1.27 (Exam Question (2007)). A curve is given by

$$
\begin{equation*}
x=t^{2}, \quad y=t e^{-t} \tag{1.2}
\end{equation*}
$$

Find $\frac{\mathrm{d} y}{\mathrm{~d} x}$ and $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$.
Where does the curve have a critical (stationary) point? Is it a maximum, minimum or point of inflection? Justify your answer.

Solution: First calculate the derivatives using the chain rule...

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{e^{-t}-t e^{-t}}{2 t}=\frac{(1-t) e^{-t}}{2 t} \\
\frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}} & =\frac{2 t\left[-e^{-t}-(1-t) e^{-t}\right]-(1-t) e^{-t}(2)}{(2 t)^{3}} \\
& =e^{-t} \frac{-2 t-2 t+2 t^{2}-2+2 t}{8 t^{3}} \\
& =\frac{e^{-t}}{4 t^{3}}\left(t^{2}-t-1\right) \\
& =\frac{e^{-t}}{4 t}-\frac{e^{-t}}{4 t^{2}}-\frac{e^{-t}}{4 t^{3}}
\end{aligned}
$$

Note that $\frac{\mathrm{d} y}{\mathrm{~d} x}=0$ only when $t=1$ (therefore it is the only possible stationary point). For the second derivative

$$
\left.\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}\right|_{t=1}=\frac{e^{-y}}{4}-\frac{e^{-y}}{4}-\frac{e^{-1}}{4}<0
$$

so our stationary point is a maximum.
Don't forget to give the Cartesian coordinates for the maximum! To do this, simply substitute $t=1$ into Equations 1.2 . You end up with:

$$
y=1 \times e^{-1}=e^{-1}, \quad x=1^{2}=1
$$

i.e. the maximum is at $\left(1, \frac{1}{e}\right)$.

### 1.5.2 Curve sketching

Thanks to modern technology, we can use graphics calculators (or even computers!) as a guide. However, you should work through the following recipe in order to accurately sketch a curve.

First let $\mathrm{y}=f(x)$. Then follow this recipe:

1) Where is $f$ defined? (Or put another way, where is it undefined?). Typically we can sometimes get vertical asymptotes.
2) Is $f$ odd or even or neither?
3) Find where $f(x)=0$ (if possible), i.e. where the curve cuts the $x$ axis.
4) Find the value of $f$ when $x=0$, i.e. $y=f(0)$, where the curve cuts the $y$ axis.
5) Find ALL stationary points and their nature (and the value of $f$ at such points)
6) Analyse the asymptotes
i. Horizontal asymptotes: What happens to $y$ as $x \rightarrow \pm \infty$ ?
ii. If $x=a$ is a vertical asymptote, what happens as $x \rightarrow a^{+}$and $x \rightarrow a^{-}$?

Note: When the notation of $x \rightarrow a^{+}$is used, this refers to the right-sided limit, i.e. $\lim _{\substack{x \rightarrow a \\ x>a}} y$. Similarly, the notation $x \rightarrow a^{-}$represents the left-sided limit $\lim _{\substack{x \rightarrow a \\ x<a}} y$.

Note 2: Often it is possible to deduce the nature of the turning point without calculating $\frac{\mathrm{d}^{2} y}{\mathrm{~d} x^{2}}$.
Example 1.28. Sketch the curve $y=f(x)=\frac{1}{x^{2}-1}$.

1) Not defined at $x= \pm 1$ (i.e. vertical asymptotes as $x= \pm 1$ ).
2) $f(-x)=f(x)$, therefore $f(x)$ is even.
3) $f(x) \neq 0$ or all $x$, therefore $f(x)$ never cuts the $x$-axis.
4) $f(0)=-1$, i.e. the curve passes through the $y$-axis at $(0,-1)$
5) For the derivative

$$
f^{\prime}(x)=-\frac{2 x}{\left(x^{2}-1\right)^{2}}=0 \quad \text { when } \quad x=0
$$

where the nature of the turning point can be determined by analysing the vertical asymptotes; you will see that $x=0$ is a maximum.

6i) For the horizontal asymptotes,

$$
\begin{gathered}
\text { As } \quad x \rightarrow \infty, \quad f(x) \rightarrow \infty \\
\text { As } \quad x \rightarrow-\infty, \quad f(x) \rightarrow \infty
\end{gathered}
$$

6ii) For the vertical asymptotes, look at $x \rightarrow 1$ first.

$$
\begin{array}{ll}
\text { As } & x \rightarrow 1^{+}, \quad f(x) \rightarrow \infty \\
\text { As } & x \rightarrow 1^{-}, \\
& f(x) \rightarrow-\infty
\end{array}
$$

and similarly for $x \rightarrow-1$,

$$
\begin{array}{ll}
\text { As } & x \rightarrow-1^{+}, \\
\text {As } & x \rightarrow-1^{-}, \\
\text {A } & f(x) \rightarrow-\infty
\end{array}
$$

At last! We are now in a position to sketch the curve; see Figure 1.4 ,


Figure 1.4: A sketch of the function $y=f(x)=1 /\left(x^{2}-1\right)$. Observe the stationary point at $x=0$; the fact that this is a maximum has been deduced with the help of the vertical asymptotes.

Example 1.29. Sketch the graph of

$$
\begin{equation*}
y^{2}=\frac{x(1-x)}{4-x^{2}} \tag{1.3}
\end{equation*}
$$

Again, we follow the recipe. . .

1) Note that

$$
y^{2}=\frac{x(1-x)}{(2-x)(2+x)},
$$

therefore there are vertical asymptotes at $x= \pm 2$. Also, are only interested in real $y$, thus we require $y^{2}>0$. Hence it follows that $y$ is defined only when

$$
\frac{x(1-x)}{4-x^{2}}>0
$$

The RHS of 1.3 may change sign at $x=0,1$, and possibly at the position of the vertical asymptotes! Consider the following diagram of the sign of $y^{2}$ :
Therefore the graph of $y$ is undefined for

$$
-2 \leq x<0 \quad \text { and } \quad 1<x \leq 2
$$

2) $y$ is neither odd nor even, but observe

$$
y= \pm \sqrt{\frac{x(1-x)}{4-x^{2}}}
$$

and the $\pm$ sign indicated that the graph should be symmetric about the horizontal $x$ axis.


Figure 1.5: You can make a sign diagram for $y^{2}=\frac{x(1-x)}{(2-x)(2+x)}$, too! Because $y^{2}$ is nonnegative for any real value of $y$, the function is undefined wherever we find that $y^{2}<0$ (these are indicated by a minus sign in the diagram).
3) $y=0$ when $x=0,1$.
4) $x=0 \quad \therefore \quad y=0$ (but we already know that!).
5) $\frac{\mathrm{d} y}{\mathrm{~d} x}$ is stationary when $\frac{\mathrm{d}}{\mathrm{d} x}\left(y^{2}\right)$ is, since $\frac{\mathrm{d}}{\mathrm{d} x}\left(y^{2}\right)=2 y \frac{\mathrm{~d} y}{\mathrm{~d} x}$.

$$
\frac{\mathrm{d}}{\mathrm{~d} x}\left(y^{2}\right)=\frac{\left(4-x^{2}\right)(1-2 x)-\left(x-x^{2}\right)(-2 x)}{\left(4-x^{2}\right)^{2}}=0
$$

For this to be zero the numerator must be zero. Therefore simplifying the numerator leads to

$$
x^{2}-8 x+4=0 \quad \therefore \quad x=4 \pm 2 \sqrt{3} \quad(\approx 0.54,7.5)
$$

Rather than calculating the second derivative (which would be quite tedious), we can deduce the nature of these turning points from the information regarding the behaviour near the horizontal asymptotes.

6i) To figure out the behaviour of the behaviour as $x \rightarrow \pm \infty$, write

$$
\begin{equation*}
y^{2}=\frac{1-\frac{1}{x}}{1-\frac{4}{x^{2}}} \tag{1.4}
\end{equation*}
$$

and use the geometric series

$$
\frac{1}{1-z}=1+z+z^{2}+\ldots, \quad \text { for } \quad|z|<1
$$

so Equation (1.4) can be approximated as (for large $|x|$ )

$$
\begin{equation*}
y^{2} \approx\left(1-\frac{1}{x}\right)\left(1+\frac{4}{x^{2}}+\ldots\right) \approx 1-\frac{1}{x} \tag{1.5}
\end{equation*}
$$

which is valid for $|x| \rightarrow \infty$. Thus

$$
\begin{array}{ll}
\text { As } & x \rightarrow \infty, \quad y \rightarrow 1^{-} \quad \text { (from below) } \\
\text { As } & x \rightarrow-\infty, \quad y \rightarrow 1^{+} \quad \text { (from above) }
\end{array}
$$

In addition, there are there are mirror images (see Step 2) of this horizontal asymptote, i.e. at $y=-1$.



Figure 1.6: Plots of the upper branch of $f(x)$ for $x<-2$ and $3<x<9$ respectively.

6ii) To get the behaviour near the vertical asymptotes it is simplest (in this case) to find where the curve cuts its horizontal asymptote, i.e. set $y^{2}=1$ :

$$
\therefore \quad 4-\not x^{2 x}=x-\not x^{22} \quad \Rightarrow \quad x=4
$$

Hence we can sketch two parts of the upper half of the graph, see Figure 1.6 .

And let's not forget to plot the rest of the graph!


Figure 1.7: The complete sketch for the (implicit) function $y^{2}=\frac{x(1-x)}{4-x^{2}}$.

### 1.5.3 Equations of Tangent and Normal

Example 1.30. Find equations of the tangent and normal to $y=x^{2}$ at $x=1$.
First find $\frac{\mathrm{d} y}{\mathrm{~d} x}$, recalling that $\frac{\mathrm{d} y}{\mathrm{~d} x} \equiv$ slope of the tangent.

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=2 x,\left.\quad \therefore \quad \frac{\mathrm{~d} y}{\mathrm{~d} x}\right|_{x=1}=2 .
$$

Also, at $x=1$ we have $y=1$. Therefore using

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

where $x_{1}=1, y_{1}=1$ and $m=2$, the line through $(1,1)$ with slope 2 has equation

$$
y=2 x-1
$$

The normal is perpendicular to the tangent. Therefore

$$
\text { Slope of Normal }=\frac{-1}{\text { Slope of Tangent }}=-\frac{1}{2}
$$

The normal is the line through $(1,1)$ with slope $=-1 / 2$. Therefore using

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

with $x_{1}=1, y_{1}=1$ and $m=-1 / 2$ yields the equation for the normal as

$$
y=-\frac{1}{2} x+\frac{3}{2}
$$



Example 1.31. Find equations of the tangent and normal to the curve given by

$$
y=t^{2}, \quad x=t^{3}+1 \quad \text { at } \quad t=1
$$

For this we use parametric differentiation

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{\frac{\mathrm{d} y}{\mathrm{~d} t}}{\frac{\mathrm{~d} x}{\mathrm{~d} t}}=\frac{2 t}{3 t^{2}}=\frac{2}{3} \quad \text { at } \quad t=1
$$

Also at $t=1,(x, y)=(2,1)$.
The tangent is the line through $(2,1)$ with slope $\frac{2}{3}$, i.e.

$$
y-1=\frac{2}{3}(x-2), \quad \therefore \quad y=\frac{2}{3} x-\frac{1}{3}
$$

The normal has slope $-\frac{3}{2}$, and thus its equation is

$$
y-1=-\frac{3}{2}(x-2), \quad \therefore \quad y=-\frac{3}{2} x+4
$$

## Chapter 2

## Hyperbolic functions

### 2.1 Definitions of hyperbolic functions

In the first chapter, we got a few glimpses of hyperbolic functions, so now you're probably itching to find out just what they are. Well, that's what this chapter is for!

First things first, here are the definitions:

$$
\begin{aligned}
\sinh x & =\frac{e^{x}-e^{-x}}{2} \\
\cosh x & =\frac{e^{x}+e^{-x}}{2} \\
\tanh x & =\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{\sinh x}{\cosh x}
\end{aligned}
$$

The three functions are pronounced "shine $x$ ", "cosh $x$ " and "tansh $x$ " respectively.
Recall that

$$
\text { as } \quad x \rightarrow \infty, \quad e^{x} \rightarrow \infty \quad \text { and } \quad e^{-x} \rightarrow 0
$$

1 If $y=\cosh x=\frac{e^{x}+e^{-x}}{2}$,

$$
\cosh (0)=1
$$

Also note that

$$
y=\cosh (-x)=\frac{e^{-x}+e^{-(-x)}}{2}=\frac{e^{-x}+e^{x}}{2}=\cosh x
$$

Therefore the curve is symmetrical about the $y$ axis, i.e. is an even function.
And

$$
\text { as } \quad x \rightarrow \infty, \quad y \rightarrow \frac{e^{x}+0}{2}=\frac{1}{2} e^{x} \rightarrow \infty .
$$

2 If $y=\sinh x=\frac{e^{x}-e^{-x}}{2}$,

$$
\sinh (0)=0
$$

Also,

$$
y=\sinh (-x)=\frac{e^{-x}-e^{-(-x)}}{2}=\frac{e^{-x}-e^{x}}{2}=-\sinh x
$$

therefore the curve is anti-symmetrical about the $y$ axis, i.e. is an odd function. And

$$
\begin{aligned}
& \text { as } \quad x \rightarrow \infty, \quad y \rightarrow \frac{e^{x}-0}{2}=\frac{1}{2} e^{x} \rightarrow+\infty, \\
& \text { as } \quad x \rightarrow-\infty, \quad y \rightarrow \frac{0-e^{-x}}{2}=-\frac{1}{2} e^{-x} \rightarrow-\infty .
\end{aligned}
$$

3 For

$$
y=\tanh x=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{\sinh x}{\cosh x}
$$

we see that

$$
\tanh (0)=\frac{0}{1}=0
$$

Also, if we consider the limits $x \rightarrow \pm \infty$ :

$$
\begin{aligned}
& \text { as } \quad x \rightarrow \infty, \quad y \rightarrow \frac{e^{x}-0}{e^{x}+0} \rightarrow 1, \\
& \text { as } \quad x \rightarrow-\infty, \quad y \rightarrow-\frac{0-e^{-x}}{0+e^{-x}} \rightarrow-1 .
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
\tanh (-x) & =\frac{\sinh (-x)}{\cosh (-x)} \\
& =\frac{-\sinh x}{\cosh x} \\
& =-\tanh x
\end{aligned}
$$

so $\tanh x$ is an odd function.


Figure 2.1: Plots of the three main hyperbolic functions. The blue curve is $\sinh x$, the red curve is $\cosh x$, and the green curve is $\tanh x$.

### 2.2 Inverse hyperbolic functions

The hyperbolic functions do come with inverse functions.

1 Suppose that

$$
y=\sinh ^{-1} x, \quad \therefore \quad x=\sinh y
$$

Then by definition,

$$
x=\frac{1}{2}\left(e^{y}-e^{-y}\right) \Longleftrightarrow \quad e^{y}-e^{-y}=2 x
$$

Multiplying by $e^{y}$ gives

$$
e^{2 y}-1-2 x e^{y}=0
$$

or

$$
\left(e^{y}\right)^{2}-2 x\left(e^{y}\right)-1=0,
$$

which is a quadratic equation in $e^{y}$.

$$
\begin{aligned}
\therefore \quad e^{y} & =\frac{2 x \pm \sqrt{4 x^{2}+4}}{2} \\
& =x \pm \sqrt{x^{2}+1}
\end{aligned}
$$

thus

$$
e^{y}=x+\sqrt{x^{2}+1}, \quad \text { or } \quad e^{y}=x-\sqrt{x^{2}+1}
$$

Now $e^{y}>0$ for all $y$, but

$$
x-\sqrt{x^{2}+1}<0
$$

because

$$
x^{2}+1>x \quad \Rightarrow \quad \sqrt{x^{2}+1}>\sqrt{x^{2}}=x
$$

So the second option (negative choice) is impossible! Hence we are left with

$$
e^{y}=x+\sqrt{x^{2}+1}
$$

or

$$
y=\sinh ^{-1} x=\ln \left(x+\sqrt{x^{2}+1}\right) .
$$

2 Suppose that

$$
y=\cosh ^{-1} x, \quad \Rightarrow \quad x=\cosh y, \quad(\text { so } \quad x \geq 1)
$$

Then by definition of cosh,

$$
\frac{1}{2}\left(e^{y}+e^{-y}\right)=x \Longleftrightarrow \quad e^{y}+e^{-y}=2 x
$$

As before, multiply by $e^{y}$ to get

$$
e^{2 y}+1-2 x e^{y}=0
$$

or

$$
\left(e^{y}\right)^{2}-2 x\left(e^{y}\right)+1=0
$$

which is a quadratic equation in $e^{y}$ (again!)

$$
\begin{aligned}
\therefore \quad e^{y} & =\frac{2 x \pm \sqrt{4 x^{2}-4}}{2} \\
& =x \pm \sqrt{x^{2}-1}
\end{aligned}
$$

and this is real since $x \geq 1$ anyway. Therefore

$$
e^{y}=x+\sqrt{x^{2}-1}, \quad \text { or } \quad e^{y}=x-\sqrt{x^{2}-1}
$$

Now $e^{y}>0$ for all $y$, and

$$
x \pm \sqrt{x^{2}-1}>0
$$

are both possibilities (so we can't rule any option out!) Observe that

$$
\begin{aligned}
\frac{1}{x+\sqrt{x^{2}-1}} & =\frac{1}{x+\sqrt{x^{2}-1}} \times \frac{x-\sqrt{x^{2}-1}}{x-\sqrt{x^{2}-1}} \\
& =\frac{x-\sqrt{x^{2}-1}}{x^{2}-\left(x^{2}-1\right)} \\
& =x-\sqrt{x^{2}-1}
\end{aligned}
$$

Thus

$$
e^{y}=x+\sqrt{x^{2}-1} \quad \text { or } \quad e^{y}=\frac{1}{x+\sqrt{x^{2}-1}}
$$

So

$$
y=\ln \left(x+\sqrt{x^{2}-1}\right)
$$

or

$$
y=\ln \left(\frac{1}{x+\sqrt{x^{2}-1}}\right)=-\ln \left(x+\sqrt{x^{2}-1}\right)
$$

i.e.

$$
y= \pm \ln \left(x+\sqrt{x^{2}-1}\right)
$$



Figure 2.2: Plot of $\cosh x$. Note that for a given value of $y$ there are two possibilities for $x$

### 2.3 Hyperbolic identities

Just like the trigonometric functions, the hyperbolic ones come with all sorts of weird and wonderful identities. You will see many of them in this section.

Now is a good time to introduce three more hyperbolic functions. They are...

$$
\begin{align*}
\operatorname{coth} x & \equiv \frac{1}{\tanh x} & (\text { c.f. } & \left.\cot x \equiv \frac{1}{\tan x}\right)  \tag{2.1}\\
\operatorname{sech} x & \equiv \frac{1}{\cosh x} & (\text { c.f. } & \left.\sec x \equiv \frac{1}{\cos x}\right)  \tag{2.2}\\
\operatorname{cosech} x & \equiv \frac{1}{\sinh x} & \text { (c.f. } & \left.\operatorname{cosec} x \equiv \frac{1}{\sin x}\right) \tag{2.3}
\end{align*}
$$

$\ldots$ and they are pronounced 'coth', 'shec' and 'coshec' respectively.
From the definitions of $\sinh x$ and $\cosh x$,

$$
\cosh x+\sinh x \equiv \frac{e^{x}+e^{-x}}{2}+\frac{e^{x}-e^{-x}}{2} \equiv e^{x}
$$

and similarly

$$
\cosh x-\sinh x \equiv \frac{e^{\not x}+e^{-x}}{2}-\frac{e^{x}-e^{-x}}{2} \equiv e^{-x}
$$

therefore

$$
(\cosh x+\sinh x)(\cosh x-\sinh x) \equiv e^{\mathscr{x}} e^{-x} \equiv 1
$$

i.e.

$$
\cosh ^{2} x-\sinh ^{2} x \equiv 1
$$

which is analogous to $\cos ^{2} x+\sin ^{2} x \equiv 1$.
Now divide the above result by $\sinh ^{2} x$ to yield

$$
\begin{aligned}
& \frac{\cosh ^{2} x}{\sinh ^{2} x}-1 \equiv \frac{1}{\sinh ^{2} x}, \\
\therefore \quad \operatorname{cosech}^{2} x & \equiv \operatorname{coth}^{2} x-1,
\end{aligned}
$$

(which is analogous to $\operatorname{cosec}^{2} x \equiv \cot ^{2} x+1$ ).
Recall that

$$
\begin{aligned}
\cosh x+\sinh x & \equiv e^{x} \\
\cosh x-\sinh x & \equiv e^{-x}
\end{aligned}
$$

Squaring both of these yields

$$
\begin{align*}
& \cosh ^{2} x+2 \sinh x \cosh x+\sinh ^{2} x \equiv e^{2 x}  \tag{2.4}\\
& \cosh ^{2} x-2 \sinh x \cosh x+\sinh ^{2} x \equiv e^{2 x} \tag{2.5}
\end{align*}
$$

and then doing (2.4) minus (2.5) yields

$$
4 \sinh x \cosh x \equiv e^{2 x}-e^{-2 x} \Longleftrightarrow 2 \sinh x \cosh x \equiv \frac{e^{2 x}-e^{-2 x}}{2}
$$

i.e.

$$
2 \sinh x \cosh x \equiv \sinh 2 x
$$

which is analogous to $2 \sin x \cos x \equiv \sin 2 x$.
But for now, let's just admire the Table 2.1. Notice that the hyperbolic identities are very similar to the trigonometric counterparts, but with some different signs! This is called Osborne's rule, which tells you to flip the sign whenever we have a product of sinhs; this includes cosech $2 x, \tanh ^{2} x$ and $\operatorname{coth}^{2} x$ as well as $\sinh ^{2} x$ ! Otherwise the hyperbolic identities are essentially the same as their trigonometric versions. You will get to derive one of these identities as part of your homework!

| Hyperbolic | Trigonometric |
| :---: | :---: |
| $\operatorname{coth} x \equiv 1 / \tanh x$ | $\cot x \equiv 1 / \tan x$ |
| $\operatorname{sech} x \equiv 1 / \cosh x$ | $\sec x \equiv 1 / \cos x$ |
| $\operatorname{cosech} x \equiv 1 / \sinh x$ | $\sec x \equiv 1 / \sin x$ |
| $\cosh ^{2} x-\sinh ^{2} x \equiv 1$ | $\cos ^{2} x+\sin ^{x} \equiv 1$ |
| $\operatorname{sech}^{2} x \equiv 1-\tanh ^{2} x$ | $\sec ^{2} x \equiv 1+\tan ^{2} x$ |
| $\operatorname{cosech}^{2} x \equiv \operatorname{coth}^{2} x-1$ | $\operatorname{cosec}^{2} x \equiv \cot ^{2} x+1$ |
| $\sinh 2 x \equiv 2 \sinh ^{2} \cosh x$ | $\sin 2 x \equiv 2 \sin x \cos x$ |
| $\cosh 2 x \equiv \cosh ^{2} x+\sinh ^{2} x$ | $\cos 2 x \equiv \cos ^{2} x-\sin ^{2} x$ |
| $\cosh 2 x \equiv 1+2 \sinh ^{2} x$ | $\cos 2 x \equiv 1-2 \sin ^{2} x$ |
| $\cosh 2 x \equiv 2 \cosh ^{2} x-1$ | $\cos 2 x \equiv 2 \cos ^{2} x-1$ |

Table 2.1: Lots of hyperbolic identities, along with with their trigonometric counterparts.

## Chapter 3

## Partial differentiation

### 3.1 Introduction to partial differentiation

Many quantities that we measure are functions of two or more variables.
Example 3.1. The temperature $T$ of a rod heated suddenly from time $t=0$ at one end.


Figure 3.1: The rod is heated at the end $x=0$. Initially, $T=0$.

Clearly $T$ depends on:
i The distance $x$ from the heated end
ii The time $t$ after heating commenced.

So we write

$$
T=T(x, t),
$$

i.e. $T$ is a function of the two independent variables: $x$ and $t$.

Example 3.2. (More abstractly), suppose that a function $f$ is defined as

$$
f(x, y)=x^{2}+3 y^{2},
$$

then the value of $f$ is determined by every possible pair $(x, y)$, so if $(x, y)=(0,2)$ then

$$
f(0,2)=0^{2}+3 \times 2^{2}=12 .
$$

Partial derivatives generalise the derivative to functions of two or more variables.

Definition 3.1. Suppose $f$ is a function of two independent variables $x$ and $y$, then the partial derivative of $f(x, y)$ w.r.t $x$ is defined as

$$
\frac{\partial f}{\partial x}=f_{x}=\lim _{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y)-f(x, y)}{\Delta x}
$$

Similarly, the partial derivative of $f(x, y)$ w.r.t $y$ is

$$
\frac{\partial f}{\partial y}=f_{y}=\lim _{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y)-f(x, y)}{\Delta y}
$$

But...there's a shortcut! If you want $f_{x}$, say, then just pretend that $y$ is a constant and differentiate with respect to $x$ only. Similarly, when you want $f_{y}$, simply pretend that $x$ is constant and go ahead with differentiating with respect to $y$ only. And yes, this lets you use (most) of the tricks we have from Chapter 1!

Example 3.3. For the function $f$ defined by

$$
f(x, y)=x^{2}+3 y^{2}
$$

find the partial derivative of $f$ w.r.t $x$ by
i Differentiating from first principles:

$$
\begin{aligned}
\frac{\partial f}{\partial x} & =\lim _{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y)-f(x, y)}{\Delta x} \\
& =\lim _{\Delta x \rightarrow 0} \frac{(x+\Delta x)^{2}+3 y^{2}-\left(x^{2}+3 y^{2}\right)}{\Delta x} \\
& =\lim _{\Delta x \rightarrow 0} \frac{2 x \Delta x+(\Delta x)^{2}}{\Delta x} \\
& =2 x
\end{aligned}
$$

ii Differentiating w.r.t $x$, treating $y$ as a constant. Then we can ignore the term $3 y^{2}$ because it vanishes, hence we end up with:

$$
\frac{\partial f}{\partial x}=2 x
$$

as above.

We can also find the partial derivative of $f$ w.r.t $y \ldots$
i Again, we use the definition:

$$
\begin{aligned}
\frac{\partial f}{\partial y} & =\lim _{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y)-f(x, y)}{\Delta y} \\
& =\lim _{\Delta y \rightarrow 0} \frac{x^{2}+3(y+\Delta y)^{2}-\left(x^{2}+3 y^{2}\right)}{\Delta y} \\
& =\lim _{\Delta y \rightarrow 0} \frac{3\left(2 y \Delta y+(\Delta y)^{2}\right)}{\Delta y} \\
& =6 y .
\end{aligned}
$$

ii Alternatively, if we differentiate $f$ w.r.t $y$, treating $x$ as a constant, we see that the $x^{2}$ term vanishes, leaving us with

$$
\frac{\partial f}{\partial y}=6 y
$$

as expected.

Physical Interpretation: Consider the heated rod problem.



Figure 3.2: Plots showing how temperature $T$ varies with respect to $t$ and to $x$ separately.
a In the top graph of Figure $3.2, \frac{\partial T}{\partial t}$ is the rate of change of $T$ with time at a fixed distance $x$.
b In bottom graph of the same figure, $\frac{\partial T}{\partial x}$ is the rate of change of $T$ with distance $x$ at a particular instance in time.

Example 3.4. Suppose

$$
f(x, y)=y \sin x+x \cos ^{2} y
$$

Then for the partial derivative $f_{x}$

$$
\frac{\partial f}{\partial x}=y \cos x+\cos ^{2} y
$$

where we treated $y$ as a constant.
Meanwhile,

$$
\begin{aligned}
\frac{\partial f}{\partial y} & =\sin x+2 x \cos y(-\sin y) \\
& =\sin x-x \sin 2 y
\end{aligned}
$$

where we treated $x$ as a constant.
Example 3.5. Suppose

$$
f(x, y)=\tan ^{-1}\left(\frac{y}{x}\right)
$$

then compute $f_{x}$ and $f_{y}$.
Recall that

$$
\frac{\mathrm{d}}{\mathrm{~d} u}\left(\tan ^{-1} u\right)=\frac{1}{1+u^{2}}
$$

Therefore, calculating $f_{x}$ (treating $y$ as a constant):

$$
f_{x}=\frac{1}{1+\left(\frac{y}{x}\right)^{2}} \frac{\partial}{\partial x}\left(\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^{2}}\left(-\frac{y}{x^{2}}\right)
$$

i.e

$$
\frac{\partial f}{\partial x}=f_{x}=-\frac{y}{x^{2}+y^{2}}
$$

Similarly, calculating $f_{y}$ (treating $x$ as a constant):

$$
f_{y}=\frac{1}{1+\left(\frac{y}{x}\right)^{2}} \frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^{2}}\left(\frac{1}{x}\right)
$$

i.e

$$
\frac{\partial f}{\partial y}=f_{y}=\frac{x}{x^{2}+y^{2}}
$$

Example 3.6 (Exam Question 2008). If a function $f(x, y)$ is defined as

$$
f(x, y)=x \ln \left(\frac{x}{y}\right)
$$

then find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
Solution: Note that

$$
f(x, y)=x \ln \left(\frac{x}{y}\right)=x(\ln x-\ln y)
$$

so for the $x$ derivative,

$$
\begin{aligned}
\frac{\partial f}{\partial x} & =1 \cdot(\ln x-\ln y)+x\left(\frac{1}{x}-0\right) \\
& =(\ln x-\ln y)+\not x \cdot \frac{1}{\not x} \\
& =\ln x-\ln y+1 \\
& =\ln \left(\frac{x}{y}\right)+1
\end{aligned}
$$

Meanwhile, for the $y$ derivative

$$
\begin{aligned}
\frac{\partial f}{\partial y} & =0-\frac{\partial}{\partial y}(x \ln y) \\
& =-x \frac{\partial}{\partial y}(\ln y) \\
& =-\frac{x}{y} .
\end{aligned}
$$

Example 3.7 (Function with three variables). Suppose $f(x, y, z)$ is defined as

$$
f(x, y, z)=z e^{y} \cos x
$$

then

$$
\begin{aligned}
\frac{\partial f}{\partial x} & =-z e^{y} \sin x \\
\frac{\partial f}{\partial y} & =z e^{y} \cos x \\
\frac{\partial f}{\partial y} & =e^{y} \cos x
\end{aligned}
$$

### 3.2 Higher Partial Derivatives

You can differentiate the first partial derivatives again to obtain second partial derivatives.

$$
\begin{aligned}
& f_{x x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^{2} f}{\partial x^{2}} \\
& f_{y y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^{2} f}{\partial y^{2}} \\
& f_{x y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^{2} f}{\partial y \partial x} \\
& f_{y x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^{2} f}{\partial x \partial y}
\end{aligned}
$$

Example 3.8. For the function

$$
f=\tan ^{-1}\left(\frac{x}{y}\right)
$$

we are given that

$$
f_{x}=\frac{y}{x^{2}+y^{2}}, \quad f_{y}=-\frac{x}{x^{2}+y^{2}} .
$$

We calculate $f_{x x}$ by treating $y$ as constant and applying the quotient rule:

$$
\begin{aligned}
f_{x x} & =\frac{\partial}{\partial x}\left[f_{x}\right]=\frac{\partial}{\partial x}\left[\frac{y}{x^{2}+y^{2}}\right] \\
& =\frac{0-y(2 x)}{\left(x^{2}+y^{2}\right)^{2}}=-\frac{2 x y}{\left(x^{2}+y^{2}\right)^{2}}
\end{aligned}
$$

In a similar fashion,

$$
\begin{aligned}
f_{y y} & =\frac{\partial}{\partial y}\left[f_{y}\right]=\frac{\partial}{\partial y}\left[\frac{-x}{x^{2}+y^{2}}\right] \\
& =\frac{0-(-x)(2 y)}{\left(x^{2}+y^{2}\right)^{2}}=\frac{2 x y}{\left(x^{2}+y^{2}\right)^{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
f_{x y} & =\frac{\partial}{\partial y}\left[f_{x}\right]=\frac{\partial}{\partial y}\left[\frac{y}{x^{2}+y^{2}}\right] \\
& =\frac{\left(x^{2}+y^{2}\right)-y(2 y)}{\left(x^{2}+y^{2}\right)^{2}} \\
& =\frac{x^{2}+y^{2}-2 y^{2}}{\left(x^{2}+y^{2}\right)^{2}}=\frac{x^{2}-y^{2}}{\left(x^{2}+y^{2}\right)^{2}}
\end{aligned}
$$

And finally,

$$
\begin{aligned}
f_{y x} & =\frac{\partial}{\partial x}\left[f_{y}\right]=\frac{\partial}{\partial x}\left[\frac{-x}{x^{2}+y^{2}}\right] \\
& =\frac{\left(x^{2}+y^{2}\right)(-1)-(-x)(2 x)}{\left(x^{2}+y^{2}\right)^{2}} \\
& =\frac{x^{2}-y^{2}}{\left(x^{2}+y^{2}\right)^{2}}=f_{x y}
\end{aligned}
$$

Fact: If $f_{x}, f_{y}, f_{x y}$ and $f_{y x}$ are continuous (i.e. doesn't 'jump') at $(x, y)$, then $f_{x y}=f_{y x}$, i.e. $f_{y x}=f_{x y}$ holds for any $f$.

Example 3.9. Let

$$
f(x, y)=x e^{2 y}
$$

$$
\begin{array}{|c|c|c|}
\hline f_{x}=e^{2 y} & f_{y}=2 x e^{2 y} & f_{y}=2 x e^{2 y} \\
f_{x y}=2 e^{2 y} & f_{y x}=2 e^{2 y} & f_{y y}=4 x e^{2 y} \\
f_{x y y}=4 e^{2 y} & f_{y x y}=4 e^{2 y} & f_{y y x}=4 e^{2 y} \\
\hline
\end{array}
$$

i.e.

$$
f_{x y y}=f_{y x y}=f_{y y x}
$$

so the order does not matter.
Example 3.10 (Exam Question 2004). a) Verify that $f(x, y)=e^{-\left(1+a^{2}\right) x} \cos a y$ is a solution of the equation

$$
\frac{\partial f}{\partial x}=\frac{\partial^{2} f}{\partial y^{2}}-f
$$

Solution: First compute the required derivatives

$$
\begin{aligned}
\frac{\partial f}{\partial x} & =-\left(1+a^{2}\right) e^{-\left(1+a^{2}\right) x} \cos a y \\
\frac{\partial f}{\partial y} & =-a e^{-\left(1+a^{2}\right) x} \sin a y \\
\frac{\partial^{2} f}{\partial y^{2}} & =-a^{2} e^{-\left(1+a^{2}\right) x} \cos a y
\end{aligned}
$$

So computing the RHS (right hand side)

$$
\begin{aligned}
\text { RHS } & =f_{y y}-f \\
& =-a^{2} e^{-\left(1+a^{2}\right) x} \cos a y-e^{-\left(1+a^{2}\right) x} \cos a y \\
& =-\left(1+a^{2}\right) e^{-\left(1+a^{2}\right) x} \cos a y=\text { LHS }
\end{aligned}
$$

b Let $g=y f(x y)$. Show that

$$
y \frac{\partial g}{\partial y}-x \frac{\partial g}{\partial x}=g
$$

## Solution:

$$
\begin{aligned}
\frac{\partial g}{\partial y} & ==f(x y)+y x f^{\prime}(x y) \\
\frac{\partial g}{\partial x} & =y^{2} f^{\prime}(x y)
\end{aligned}
$$

where primes denote differentiation w.r.t the combined variable $x y$.

Note: To see this, consider

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(\sin 2 x)=2 \cos 2 x
$$

i.e

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(f(2 x))=2 f^{\prime}(2 x) .
$$

Also consider

$$
\frac{\partial}{\partial x}(\sin x y)=y \cos x y
$$

and therefore

$$
\frac{\partial}{\partial x}(f(x y))=y f^{\prime}(x y)
$$

Hence returning to the example,

$$
\mathrm{LHS}=y f(x y)+x y^{2} f^{\prime}(x y)-x y^{2} f^{\prime}(x y)=g(x, y)=\text { RHS },
$$

as required.

## Chapter 4

## Integration

### 4.1 The basics

There are two ways to interpret integration...

1. Integration is the reverse of differentiation! If we have, say,

$$
\frac{\mathrm{d} \mathcal{A}(x)}{\mathrm{d} x}=f(x)
$$

then we can write

$$
\mathcal{A}(x)=\int f(x) \mathrm{d} x+C . \quad[\text { Indefinite integral! }]
$$

We say that $\mathcal{A}$ is the integral (antiderivative) of $f(x)$.
2. Integration gives the area under a curve To achieve this, you sum the contribution of lots of infinitesimally small pieces.

To demonstrate, consider the area bounded by the $x$-axis, the lines $x=a, x=b$ and the curve $y=f(x)$, as shown in the following diagram:


It is often taken for granted that the two interpretations are the same. In fact, this is not obvious, so mathematicians have a big theorem about it...

Theorem: Fundamental Theorem of Calculus
The shaded area above is

$$
\int_{a}^{b} f(x) \mathrm{d} x .
$$



Proof: Let $\mathcal{A}(x)=$ area from say, the origin $O$ to the point $x$ under the curve. Then the area of the shaded rectangle is

$$
\mathcal{A}(x+h)-\mathcal{A} \approx f(x) h .
$$

[Note: The intuition behind the above approximation is that it becomes more accurate as $h \rightarrow 0$ !]

$$
\therefore \quad f(x) \approx \frac{\mathcal{A}(x+h)-\mathcal{A}(x)}{h} \rightarrow \frac{\mathrm{~d} A(x)}{\mathrm{d} x} \quad \text { as } h \rightarrow 0 .
$$

Therefore the area from $x=a$ to $x=b$ is

$$
\mathcal{A}(b)-\mathcal{A}(a)=\int_{a}^{b} f(x) \mathrm{d} x . \quad \text { [A number; a definite integral!! }
$$

When tackling an integral, an engineer can count on these standard results...

| $f(x)$ | $\int f(x) \mathrm{d} x$ |
| :---: | :---: |
| $x^{n}(n \neq-1)$ | $\frac{1}{n+1} x^{n+1}+C$ |
| $x^{-1}$ | $\ln \|x\|+C$ |
| $e^{a x}$ | $\frac{1}{a} e^{a x}+C$ |
| $\cos (a x)$ | $\frac{1}{a} \sin (a x)+C$ |
| $\sin (a x)$ | $-\frac{1}{a} \cos (a x)+C$ |
| $\frac{1}{x^{2}+1}$ | $\tan ^{-1} x+C$ |

Table 4.1: Table of Basic Integrals

### 4.2 Integration by substitution

Sometimes an integral is easier to solve if you change the variable you are integrating with respect to, i.e. make a substitution.

$$
\begin{gathered}
\text { Formally, if } \quad I=\int_{x_{1}}^{x_{2}} f(x) \mathrm{d} x, \\
\text { try introducing } \quad u=g(x), \\
\Rightarrow \quad \frac{\mathrm{d} u}{\mathrm{~d} x}=g^{\prime}(x) \quad \text { or } \quad \frac{\mathrm{d} x}{\mathrm{~d} u}=\frac{1}{g^{\prime}(x)},
\end{gathered}
$$

so we end up with something that looks like multiplying and dividing by $\mathrm{d} u$ :

$$
I=\int_{x_{1}}^{x_{2}} f(x) \mathrm{d} x=\int_{u_{1}}^{u_{2}} f(u) \frac{\mathrm{d} x}{\mathrm{~d} u} \mathrm{~d} u
$$

where $u_{1}=g\left(x_{1}\right), u_{2}=g\left(x_{2}\right)$. So you must change the upper and lower limits for your definite integral.

The best time to use this is when you have a function "wrapped" in another function you would like to unravel.

Example 4.1. Calculate the integral

$$
\int(3 x-7)^{-5} \mathrm{~d} x
$$

We want to remove the "function of a function", so let

$$
u=3 x-7 \quad \Rightarrow \quad \mathrm{~d} u=3 \mathrm{~d} x \quad \Rightarrow \quad \mathrm{~d} x=\frac{1}{3} \mathrm{~d} u
$$

then

$$
\begin{aligned}
\int(3 x-7)^{-5} \mathrm{~d} x & =\frac{1}{3} \int u^{-5} \mathrm{~d} u \\
& =\frac{1}{3}\left(-\frac{1}{4} u^{-4}\right)+C \\
& =-\frac{1}{12} u^{-4}+C \\
& =-\frac{1}{12}(3 x-7)^{-4}+C .
\end{aligned}
$$

Don't forget to rewrite your final answer in terms of $x$ !
Example 4.2. Calculate the integral

$$
\int \frac{\sin \sqrt{x}}{\sqrt{x}} \mathrm{~d} x .
$$

Here, the 'horrible' bit is $\sqrt{x}$, so let

$$
u=\sqrt{x} \quad \Rightarrow \quad \mathrm{~d} u=\frac{1}{2 \sqrt{x}} \mathrm{~d} x
$$

i.e.

$$
\begin{aligned}
& \mathrm{d} x=2 \sqrt{x} \mathrm{~d} u=2 u \mathrm{~d} u \\
& \int \frac{\sin \sqrt{x}}{\sqrt{x}} \mathrm{~d} x=\int \frac{\sin u}{\not x} \cdot 2 \not x \mathrm{~d} u \\
&=2 \int \sin u \mathrm{~d} u \\
&=-2 \cos u+C \\
&=-2 \cos \sqrt{x}+C
\end{aligned}
$$

## Example 4.3.

$$
\mathscr{I}=\int \sqrt{x}(1+\sqrt{x})^{\frac{1}{4}} \mathrm{~d} x
$$

If we let $u=\sqrt{x}$ we still end up with a term that looks like $u^{2}(1+u)^{\frac{1}{4}}$ which is still difficult to deal with.

How about. . . $u=1+\sqrt{x}$ ?

$$
\mathrm{d} u=\frac{1}{2 \sqrt{x}} \mathrm{~d} x \quad \Rightarrow \quad \mathrm{~d} x=2 \sqrt{x} \mathrm{~d} u=2(u-1)=2 \sqrt{x} \mathrm{~d} u
$$

Subsequently,

$$
\begin{aligned}
\int \sqrt{x}(1+\sqrt{x})^{\frac{1}{4}} \mathrm{~d} x & =\int(u-1) u^{\frac{1}{4}} 2(u-1) \mathrm{d} u \\
& =2 \int(u-1)^{2} u^{\frac{1}{4}} \mathrm{~d} u \\
& =2 \int u^{\frac{1}{4}}\left(u^{2}-2 u+1\right) \mathrm{d} u \\
& =2\left(\frac{4}{13} u^{\frac{13}{4}}-2 \frac{4}{9} u^{\frac{9}{4}}+\frac{4}{5} u^{\frac{5}{4}}\right)+C \\
& =\frac{8}{13}(1+\sqrt{x})^{\frac{13}{4}}-\frac{16}{9}(1+\sqrt{x})^{\frac{9}{4}}+\frac{8}{5}(1+\sqrt{x})^{\frac{5}{4}}+C .
\end{aligned}
$$

### 4.2.1 A question of logs

Let us consider the derivative of the logarithm of some general function $f(x)$ :

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} x}(\ln (f(x))) & =\frac{1}{f(x)} \cdot \frac{\mathrm{d}}{\mathrm{~d} x}(f(x)) \\
& =\frac{f^{\prime}(x)}{f(x)}
\end{aligned}
$$

This implies that:

$$
\int \frac{f^{\prime}(x)}{f(x)} \mathrm{d} x=\ln (f(x))+c
$$

Example 4.4. Consider the the following integral:

$$
\mathscr{I}=\int \frac{2 x+5}{x^{2}+5 x+3} \mathrm{~d} x
$$

Now, if we choose $f(x)=x^{2}+5 x+3$, then $f^{\prime}(x)=2 x+5$. So, if we differentiate $\ln (f(x))$, in this case we have

$$
\frac{d}{d x}\left[\ln \left(x^{2}+5 x+3\right)\right]=\frac{2 x+5}{x^{2}+5 x+3}
$$

by the chain rule. Thus we know the integral must be

$$
\mathscr{I}=\ln \left(x^{2}+5 x+3\right)+C .
$$

### 4.2.2 Trigonometric and hyperbolic substitutions

| If you see | Try substituting |
| :---: | :---: |
| $\sqrt{a^{2}-x^{2}}$ | $x=a \sin \theta$ |
| $\sqrt{a^{2}+x^{2}}$ | $x=a \sinh \theta$ |
| $\sqrt{x^{2}-a^{2}}$ | $x=a \cosh \theta$ |
| $\frac{1}{a^{2}+x^{2}}$ | $x=a \tan \theta$ |

Example 4.5 (To show why).

$$
\mathscr{I}=\int \frac{1}{\sqrt{a^{2}+x^{2}}} \mathrm{~d} x
$$

If we let $x=a \sinh \theta$, then

$$
\mathrm{d} x=a \cosh \theta \mathrm{~d} \theta
$$

thus

$$
\begin{aligned}
\mathscr{I} & =\int \frac{a \cosh \theta}{\sqrt{a^{2}+a^{2} \sinh ^{2} \theta}} \mathrm{~d} \theta \\
& =\int \frac{\not \alpha \cosh \theta}{\not a \sqrt{1+\sinh ^{2} \theta}} \mathrm{~d} \theta \\
& =\int \frac{\cosh \theta}{\cosh \theta} \mathrm{d} \theta \\
& =\int 1 \mathrm{~d} \theta \\
& =\theta+C=\sinh ^{-1}\left(\frac{x}{a}\right) .
\end{aligned}
$$

Example 4.6 (Harder!).

$$
\begin{aligned}
\mathscr{I} & =\int_{-3}^{-1} \frac{1}{\sqrt{14-12 x-2 x^{2}}} \mathrm{~d} x \\
& =\frac{1}{\sqrt{2}} \int_{-3}^{-1} \frac{1}{\sqrt{7-6 x-x^{2}}} \mathrm{~d} x
\end{aligned}
$$

Not obvious what the next step is.
Complete the square in the denominator!

$$
7-6 x-x^{2}=7-(x+3)^{2}+9=16-(x+3)^{2}
$$

Hence

$$
\mathscr{I}=\frac{1}{\sqrt{2}} \int_{-3}^{-1} \frac{1}{\sqrt{16-(x+3)^{2}}} \mathrm{~d} x
$$

which looks like

$$
\frac{1}{\sqrt{a^{2}-u^{2}}}
$$

so we will choose a substitution like $a \sin \theta$.

Let $u=x+3$, then $\mathrm{d} u=\mathrm{d} x$, and as a result:

$$
\mathscr{I}=\frac{1}{\sqrt{2}} \int_{0}^{2} \frac{1}{\sqrt{16-u^{2}}} \mathrm{~d} u
$$

Now put

$$
\begin{aligned}
u & =4 \sin \theta \Rightarrow \mathrm{~d} u=4 \cos \theta \mathrm{~d} \theta \\
\mathscr{I} & =\frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{6}} \frac{4 \cos \theta}{\sqrt{16-16 \sin ^{2} \theta}} \mathrm{~d} \theta \\
& =\frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{6}} \frac{4 \cos \theta}{4 \cos \theta} \mathrm{~d} \theta \\
& =\frac{1}{\sqrt{2}} \int_{0}^{\frac{\pi}{6}} 1 \mathrm{~d} \theta \\
& =\frac{\pi}{6 \sqrt{2}} \\
& =\frac{\pi \sqrt{2}}{12}
\end{aligned}
$$

### 4.2.3 One more trick

If you see an integral like

$$
\int \sin ^{4} x \cos x \mathrm{~d} x
$$

try $u=\sin x$, because you get $\mathrm{d} u=\cos x \mathrm{~d} x$, making the cos term disappear.
However, if you are facing

$$
\int \sin ^{4} x \cos ^{3} x \mathrm{~d} x
$$

keep your eyes open for less obvious clues!

$$
\begin{aligned}
& =\int \sin ^{4} x \cos ^{2} x \cos x \mathrm{~d} x \\
& =\int \sin ^{4} x\left(1-\sin ^{2} x\right) \cos x \mathrm{~d} x \\
& =\int \sin ^{4} x \cos x \mathrm{~d} x-\int \sin ^{6} x \cos x \mathrm{~d} x
\end{aligned}
$$

then we can summon $u=\sin x$.

Remark 4.1. This even works for, say,

$$
\int \cos ^{5} x \mathrm{~d} x=\int\left(1-\sin ^{2} x\right)^{2} \cos x \mathrm{~d} x
$$

And finally... be bold! Try!

### 4.3 Integration by parts

This is a good strategy when you are integrating a product of two terms, one of which either differentiates or integrates into something simpler.

Recall the product rule:

$$
\frac{\mathrm{d}}{\mathrm{~d} x}(u v)=v \frac{\mathrm{~d} u}{\mathrm{~d} x}+u \frac{\mathrm{~d} v}{\mathrm{~d} x}
$$

Now integrate both sides w.r.t. $x$ :

$$
\begin{aligned}
u v & =\int v \frac{\mathrm{~d} u}{\mathrm{~d} x} \mathrm{~d} x+\int u \frac{\mathrm{~d} v}{\mathrm{~d} x} \mathrm{~d} x \\
\Rightarrow \quad \int u \frac{\mathrm{~d} v}{\mathrm{~d} x} \mathrm{~d} x & =u v-\underbrace{\int v \frac{\mathrm{~d} u}{\mathrm{~d} x} \mathrm{~d} x}_{\text {Another integral! }},
\end{aligned}
$$

The idea is that $u$ becomes "better" as you differentiate or $\frac{\mathrm{d} v}{\mathrm{~d} x}$ becomes "better" as you integrate.

## Example 4.7. Find

$$
\int x e^{x} \mathrm{~d} x .
$$

Since $x$ differentiates away nicely,

$$
\begin{gathered}
\text { choose } \quad u=x, \quad \frac{\mathrm{~d} v}{\mathrm{~d} x}=e^{x}, \\
\text { then } \frac{\mathrm{d} u}{\mathrm{~d} x}=1, \quad v=\int e^{x} \mathrm{~d} x=e^{x} .
\end{gathered}
$$

Apply the by parts formula:

$$
\begin{align*}
\int x e^{x} \mathrm{~d} x & =x e^{x}-\int 1 \cdot e^{x} \mathrm{~d} x \\
& =x e^{x}-e^{x}+C . \\
& =e^{x}(x-1)+C . \tag{4.1}
\end{align*}
$$

(Note that the arbitrary constant has been included right at the very last step)
Question: What happens if you try the other way round?

$$
\text { If } \quad u=e^{x}, \quad \frac{\mathrm{~d} v}{\mathrm{~d} x}=x,
$$

$$
\text { then } \frac{\mathrm{d} u}{\mathrm{~d} x}=e^{x}, \quad v=\frac{x^{2}}{2},
$$

which already does not look promising. If we go ahead and use the by-parts rule, then...

$$
\int x e^{x} \mathrm{~d} x=\frac{x^{2}}{2} e^{x}-\frac{1}{2} \int x^{2} e^{x} \mathrm{~d} x
$$

which is true, but does not help!

So what have we learned from this example? Well, it does matter which term you choose for $u$ or $\frac{\mathrm{d} v}{\mathrm{~d} x}$, as it can make or break your hopes of solving an integral. So choose wisely!

Example 4.8. Find

$$
\mathscr{I}=\int e^{2 x} \sin x \mathrm{~d} x .
$$

Let

$$
\begin{gathered}
u=\sin x, \quad \frac{\mathrm{~d} v}{\mathrm{~d} x}=e^{2 x}, \\
\text { then } \quad \frac{\mathrm{d} u}{\mathrm{~d} x}=\cos x, \quad v=\frac{1}{2} e^{2 x}
\end{gathered}
$$

and the by-parts formula gives:

$$
\begin{aligned}
\mathscr{I} & =\frac{1}{2} e^{2 x} \sin x-\frac{1}{2} \int e^{2 x} \cos x \mathrm{~d} x \\
& =\frac{1}{2} e^{2 x} \sin x-\frac{1}{2} \mathscr{J},
\end{aligned}
$$

where

$$
\mathscr{J}=\int e^{2 x} \cos x \mathrm{~d} x,
$$

yet another integral. But don't panic! This one can be handled by parts too; simply let

$$
\begin{gathered}
u=\cos x, \quad \frac{\mathrm{~d} v}{\mathrm{~d} x}=e^{2 x}, \\
\text { then } \frac{\mathrm{d} u}{\mathrm{~d} x}=-\sin x, \quad v=\frac{1}{2} e^{2 x},
\end{gathered}
$$

which gives

$$
\begin{aligned}
\mathscr{J} & =\frac{1}{2} e^{2 x} \cos x+\frac{1}{2} \int e^{2 x} \sin x \mathrm{~d} x \\
& =\frac{1}{2} e^{2 x} \cos x+\frac{1}{2} \mathscr{I} . \\
\therefore \quad \mathscr{I} & =\frac{1}{2} e^{2 x} \sin x-\frac{1}{4}\left(e^{2 x} \cos x+\mathscr{I}\right) \\
\Rightarrow \quad \frac{5}{4} \mathscr{I} & =\frac{1}{2} e^{2 x} \sin x-\frac{1}{4} e^{2 x} \cos x,
\end{aligned}
$$

So, finally, we have:

$$
\therefore \quad \mathscr{I}=\frac{1}{5}\left(2 e^{2 x} \sin x-e^{2 x} \cos x\right)+C,
$$

not forgetting the constant of integration at the very end!

Example 4.9. Compute

$$
\begin{aligned}
& \int \ln x \mathrm{~d} x . \quad \text { (Classic A-Level question!) } \\
& \begin{aligned}
\int \ln x \mathrm{~d} x & =\int 1 \cdot \ln x \mathrm{~d} x \\
& =x \ln x-\int \not x x^{\frac{1}{x}} \mathrm{~d} x \\
& =x(\ln x-1)+C
\end{aligned}
\end{aligned}
$$

Example 4.10. Find

$$
\begin{gathered}
\mathscr{I}=\int \sin ^{-1} x \mathrm{~d} x . \\
\mathscr{I}=\int 1 \cdot \sin ^{-1} x \mathrm{~d} x \\
=x \sin ^{-1} x-\int \frac{x}{\sqrt{1-x^{2}}} \mathrm{~d} x \\
= \\
x \sin ^{-1} x-\sqrt{1-x^{2}} .
\end{gathered}
$$

### 4.4 Using partial fractions

Sometimes we want to compute, say,

$$
\int \frac{x+1}{x^{2}-3 x+2} \mathrm{~d} x
$$

which we can't integrate directly. Here we must express the integrand as a sum of partial fractions.

### 4.4.1 Recap: Partial fractions

You can express the function $\frac{P(x)}{Q(x)}$ with partial fractions if $Q(x)$ factorises.

| For every factor of $Q(x)$ | You get this partial fraction form: |
| :---: | :---: |
| $(a x+b)$ | $\frac{A}{(a x+b)}$ |
| $(a x+b)^{2}$ | $\frac{A}{(a x+b)}+\frac{B}{(a x+b)^{2}}$ |
| $(a x+b)^{3}$ | $\frac{A}{(a x+b)}+\frac{B}{(a x+b)^{2}}+\frac{C}{(a x+b)^{3}}$ |
| $\left(a x^{2}+b x+c\right)$ | $\frac{A x+B}{a x^{2}+b x+c}$ |

Then plug in some different values of $x$ to find $A, B, \ldots$ (or use any other method you prefer!)

For the next three examples $P(x)$ will be linear and $Q(x)$ will be quadratic polynomials.

Example 4.11 (Case 1: Denominator has two real roots).

$$
\int \frac{3 x-5}{x^{2}-2 x-3} \mathrm{~d} x
$$

First things first. . . factorise the denominator!

$$
\begin{aligned}
x^{2}-2 x-3 & \equiv(x-3)(x+1) \\
\therefore \quad & \frac{3 x-5}{x^{2}-2 x-3} \equiv \frac{A}{(x-3)}+\frac{B}{x+1} .
\end{aligned}
$$

Hence

$$
3 x-5 \equiv A(x+1)+B(x-3)
$$

Let's try two different values of $x$. How about. . . ?

$$
\begin{gathered}
x=-1 \Rightarrow-8=-4 B \Rightarrow B=2 \\
x=3 \Rightarrow 4=4 A \Rightarrow A=1 \\
\therefore \quad \frac{3 x-5}{x^{2}-2 x-3} \equiv \frac{1}{(x-3)}+\frac{2}{x+1} .
\end{gathered}
$$

Then

$$
\begin{aligned}
& \int \frac{3 x-5}{x^{2}-2 x-3} \mathrm{~d} x \\
= & \int\left(\frac{1}{x-3}+\frac{2}{x+1}\right) \mathrm{d} x \\
= & \int \frac{1}{x-3} \mathrm{~d} x+\int \frac{2}{x+1} \mathrm{~d} x \\
= & \ln |x-3|+2 \ln |x+1|+C .
\end{aligned}
$$

Example 4.12 (Case 2: Denominator has one real root).

$$
\int \frac{x}{x^{2}-2 x+1} \mathrm{~d} x
$$

Start with

$$
\begin{gathered}
\frac{x}{x^{2}-2 x+1} \equiv \frac{x}{(x-1)^{2}} \equiv \frac{A}{x-1}+\frac{B}{(x-1)^{2}} \\
\therefore \quad x \equiv A(x-1)+B \equiv A x+B-A
\end{gathered}
$$

Let's compare coefficients: the $x$ terms suggest that $A=1$. As for the constant terms:

$$
B-A=0 \Rightarrow A=B=1
$$

Therefore

$$
\begin{aligned}
& \int \frac{x}{x^{2}-2 x+1} \mathrm{~d} x \\
= & \int \frac{1}{x-1} \mathrm{~d} x+\int \frac{1}{(x-1)^{2}} \mathrm{~d} x \\
= & \ln |x-1|-\frac{1}{x-1}+C .
\end{aligned}
$$

Example 4.13 (Case 3: Denominator has no real roots).

$$
\int \frac{x-2}{x^{2}-2 x+5} \mathrm{~d} x
$$

So we can't factorise the denominator, but we can still complete the square!

$$
x^{2}-2 x+5=(x-1)^{2}+4
$$

thus the integral is

$$
\int \frac{x-2}{(x-1)^{2}+4} \mathrm{~d} x
$$

Looks like something with $\left(u^{2}+1\right)$, so choose

$$
x-1=u, \quad \Rightarrow \quad \mathrm{~d} x=\mathrm{d} u
$$

Then

$$
\begin{aligned}
\int \frac{x-2}{x^{2}-2 x+5} \mathrm{~d} x & =\int \frac{u-1}{u^{2}+4} \mathrm{~d} u \\
& =\int \frac{u}{u^{2}+4} \mathrm{~d} u-\int \frac{1}{u^{2}+4} \mathrm{~d} u
\end{aligned}
$$

Now

$$
\begin{aligned}
\int \frac{u}{u^{2}+4} \mathrm{~d} u & =\frac{1}{2} \ln \left|u^{2}+4\right| \\
& =\frac{1}{2} \ln \left|(x-1)^{2}+4\right|
\end{aligned}
$$

while for the other $u$-integral, try

$$
u=2 \tan \theta \quad \Rightarrow \quad \mathrm{~d} u=2 \sec ^{2} \theta \mathrm{~d} \theta
$$

hence

$$
\begin{aligned}
\int \frac{1}{u^{2}+4} \mathrm{~d} u & =\int \frac{2 \sec ^{2} \theta}{4 \tan ^{2} \theta+4} \mathrm{~d} \theta \\
& =\int \frac{\sec ^{2} \theta}{2 \sec ^{2} \theta} \mathrm{~d} \theta \\
& =\int \frac{1}{2} \mathrm{~d} \theta \\
& =\frac{1}{2} \theta+C=\frac{1}{2} \tan ^{-1}\left(\frac{x-1}{2}\right)+C
\end{aligned}
$$

Thus our final answer is

$$
\int \frac{x-2}{x^{2}-2 x+5} \mathrm{~d} x=\frac{1}{2} \ln \left(x^{2}-2 x+5\right)+\frac{1}{2} \tan ^{-1}\left(\frac{x-1}{2}\right)+C .
$$

Remark 4.2. If degree of $P \geq$ degree of $Q$, use long division first to get $N(x)+\frac{R(x)}{Q(x)}(R$ for remainder!). Then use partial fractions on $\frac{R(x)}{Q(x)}$.

Example 4.14. Evaluate the indefinite integral

$$
\int \frac{x^{3}+2 x}{x-1} \mathrm{~d} x
$$

Do the long division first:

$$
x-1) \begin{array}{r}
x^{2}+x+3 \\
\frac{x^{3}+2 x}{-x^{3}+x^{2}} x^{2}+2 x \\
\frac{-x^{2}+x}{3 x} \\
\frac{-3 x+3}{3}
\end{array}
$$

$$
\begin{aligned}
\therefore \int \frac{x^{3}+2 x}{x-1} \mathrm{~d} x & =\int\left(x^{2}+x+3+\frac{3}{x-1}\right) \mathrm{d} x \\
& =\frac{x^{3}}{3}+\frac{x^{2}}{2}+3 x+3 \log |x-1|+C .
\end{aligned}
$$

### 4.5 Some trigonometric integrals

i Evaluate

$$
\begin{aligned}
\int \cos ^{2} x \mathrm{~d} x & =\int \frac{1}{2}(\cos 2 x+1) \mathrm{d} x \\
& =\frac{1}{4} \sin 2 x+\frac{1}{2} x+C
\end{aligned}
$$

ii Evaluate

$$
\begin{aligned}
\int \sin ^{2} x \mathrm{~d} x & =\int \frac{1}{2}(1-\cos 2 x) \mathrm{d} x \\
& =\frac{1}{2} x-\frac{1}{4} \sin 2 x+C
\end{aligned}
$$

### 4.6 Using integration

As stated at the start of the chapter, integration is great for calculating areas under curves.
Example 4.15 (1997 Exam question). Sketch the region enclosed by the curve $y=\frac{1}{1+x^{2}}$ and the line $y=\frac{1}{2}$ and find its area.

Apply the recipe for curve sketching:

- No vertical asymptotes
- An even function
- Passes through $(0,1)$
- $y \neq 0$, and in fact $y>0$ for all $x$.
- $y \rightarrow 0$ as $x \rightarrow \pm \infty$.
- For the turning points

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=-\frac{2 x}{\left(1+x^{2}\right)^{2}}=0 \quad \text { when } \quad x=0 .
$$

Now don't forget the sketch!


Figure 4.1: A sketch of the curve $y=\frac{1}{1+x^{2}}$ (red) and the line $y=\frac{1}{2}$ (yellow). The enclosed region is shaded in green.

$$
\begin{aligned}
\mathcal{A} & =\int_{-1}^{1} \frac{1}{1+x^{2}} \mathrm{~d} x-\text { (Area of Rectangle) } \\
& =\int_{-1}^{1} \frac{1}{1+x^{2}} \mathrm{~d} x-2 \times \frac{1}{2} \\
& =\left[\tan ^{-1} x\right]_{-1}^{1}-1 \\
& =\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)-1=\frac{\pi}{2}-1
\end{aligned}
$$

Example 4.16. Question: Find the area bounded by the curve $y=x^{2}-6 x+5$ and the
$x$ axis between $x=1$ and $x=3$.

$$
\begin{aligned}
\mathcal{A} & =\int_{1}^{3} y \mathrm{~d} x=\int_{1}^{3}\left(x^{2}-6 x+5\right) \mathrm{d} x \\
& =\left[\frac{1}{3} x^{3}-3 x^{2}+5 x\right]_{1}^{3} \\
& =\left[\frac{1}{3} \cdot 3^{3}-3 \cdot 3^{2}+5 \cdot 3\right]-\left[\frac{1}{3} \cdot 1^{3}-3 \cdot 1^{2}+5 \cdot 1\right] \\
& =-5 \frac{1}{3}
\end{aligned}
$$

But why is the area negative? Let's draw a sketch.


Figure 4.2: $A$ sketch of the curve $y=x^{2}-6 x+5$ (red). The region we want to integrate over (blue) is bounded by the grey vertical lines $x=1$ and $x=3$. Trouble is, the region below the $x$ axis gives a negative area!

Example 4.17. (Mechanics)
A ball is thrown down from a high building with an initial velocity of 30 metres per second. Then its velocity after $t$ seconds is given by $v(t)=10 t+30$. How far does the ball fall between 1 and 3 seconds of elapsed time?

The distance $s(t)$ turns out to be the integral of the velocity, i.e.

$$
s(t)=\int v(t) \mathrm{d} t
$$

Hence the distance we want is

$$
\begin{aligned}
s(3)-s(1) & =\int_{1}^{3} v(t) \mathrm{d} t \\
& =\int_{1}^{3}(10 t+30) \mathrm{d} t \\
& =\left[5 t^{2}+30 t\right]_{1}^{3} \\
& =135-35 \\
& =100 \text { metres. }
\end{aligned}
$$

Example 4.18. Find the area $\mathcal{A}$ of an ellipse, given by the equation

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1
$$



Figure 4.3: An ellipse
Note from Figure 4.3 that $\mathcal{A}=4 \times A_{1}$ by symmetry. Hence for the area $\mathcal{A}$,

$$
\begin{aligned}
\mathcal{A} & =4 \int_{0}^{a} b \sqrt{1-\frac{x^{2}}{a^{2}}} \mathrm{~d} x \\
& =4 b \int_{0}^{a} \sqrt{1-\frac{x^{2}}{a^{2}}} \mathrm{~d} x
\end{aligned}
$$

an integral that can be solved by substitution. Let

$$
\frac{x}{a}=\sin u, \quad \Rightarrow \quad \frac{\mathrm{~d} x}{\mathrm{~d} u}=a \cos u
$$

and

$$
\sqrt{1-\frac{x^{2}}{a^{2}}}=\sqrt{1-\sin ^{2} u}=\cos u
$$

So we have

$$
\mathcal{A}=4 b \int_{u_{1}}^{u_{2}} \cos u(a \cos u) \mathrm{d} u .
$$

Reminder: In changing the variable it is also very important to change the limits, i.e. find numerical values for $u_{1}$ and $u_{2}$.

$$
\begin{aligned}
& \text { When } \quad x=a, \quad \sin u=1, \quad \therefore \quad u=\frac{\pi}{2} \\
& \text { When } \quad x=0, \quad \sin u=0, \quad \therefore \quad u=0 .
\end{aligned}
$$

Therefore we have

$$
\mathcal{A}=4 a b \int_{0}^{\frac{\pi}{2}} \cos ^{2} u \mathrm{~d} u
$$

Proceeding with the integral, we get

$$
\begin{aligned}
\mathcal{A} & =4 a b \int_{0}^{\frac{\pi}{2}} \cos ^{2} u \mathrm{~d} u \\
& =4 a b \int_{0}^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{2} \cos 2 u\right) \mathrm{d} u \\
& =4 a b\left(\frac{1}{2} u+\frac{1}{4} \sin 2 u\right) \\
& =4 a b\left(\frac{\pi}{4}+0-(0+0)\right) \\
& =\pi a b
\end{aligned}
$$

Note: For a circle, $a=b$ which givws $\underline{\mathcal{A}=\pi a^{2}}$.

### 4.7 Improper integrals

Often, you will come across integrals of the type

$$
\int_{a}^{\infty} f(x) \mathrm{d} x
$$

This is an improper integral, and it must be interpreted as

$$
=\lim _{b \rightarrow \infty} \int_{a}^{b} f(x) \mathrm{d} x
$$

if the limit exists! (If it doesn't, the integral is said to diverge).
Remark 4.3. Technically, there are other kinds of improper integrals, in which

$$
\mathscr{I}=\int_{a}^{b} f(x) \mathrm{d} x
$$

has a problem because $f(x)$ "blows up" at $a$, $b$ or some point $c$ in between ( $a<c<b$ ). But we won't worry about them here!

Example 4.19. Consider

$$
\mathscr{I}=\int_{1}^{\infty} \frac{1}{x^{n}} \mathrm{~d} x, \quad n>1
$$

Then

$$
\begin{aligned}
\int_{1}^{\infty} \frac{1}{x^{n}} \mathrm{~d} x & =\lim _{b \rightarrow \infty} \int_{1}^{b} \frac{1}{x^{n}} \mathrm{~d} x \\
& =\lim _{b \rightarrow \infty}\left(\frac{1}{n-1}\left[1-\frac{1}{b^{n-1}}\right]\right) \\
& =\frac{1}{n-1}
\end{aligned}
$$

Remark 4.4. This integral in this last example diverges for $n \leq-1$.

## Chapter 5

## Differential Equations

### 5.1 Introduction

Many problems in engineering and physical science (also biology, economics, etc.) can be reduced to solving differential equations.

Example 5.1 (RLC Series Circuit). Consider the following series circuit comprised of a resistor, a capacitor and an inductor. This circuit is known as an RLC circuit.


Figure 5.1: An RLC Circuit

$$
\begin{equation*}
L \frac{\mathrm{~d}^{2} I}{\mathrm{~d} t^{2}}+R \frac{\mathrm{~d} I}{\mathrm{~d} t}+\frac{1}{C} I=E \tag{5.1}
\end{equation*}
$$

where

$$
\begin{aligned}
I & \equiv \text { Current Flowing in a Circuit } \\
C & \equiv \text { Capacitance } \\
R & \equiv \text { Resistance } \\
L & \equiv \text { Inductance } \\
E & \equiv \text { Voltage. }
\end{aligned}
$$

where $C, R, L$ and $E$ are constants and $I$ is the unknown function to be found.

An ordinary differential equation $(O D E)$ is a relation between a function $y(x), x$, and the derivatives $\frac{\mathrm{d} y}{\mathrm{~d} x}, \frac{\mathrm{~d}^{2} y}{\mathrm{~d} x^{2}}$, etc.

The order of the ODE is the order of the highest derivative in the equation.
An ODE is linear if there are no products of $y$ and its derivatives, e.g.

$$
y \frac{\mathrm{~d} y}{\mathrm{~d} x}, \quad y^{2}
$$

and no functions of $y$ and its derivatives, such as

$$
e^{y}, \quad \cos y
$$

For example, Equation (5.1) is a linear second order ode.
Example 5.2 (Legendre's Equation).

$$
\left(1-x^{2}\right) y^{\prime \prime}-2 x y^{\prime}+k^{2} y=0 \quad(k=\text { constant })
$$

is ubiquitous in problems with spherical symmetry (e.g a Hydrogen atom). It is a linear second order equation.

Example 5.3 (Radioactive decay).

$$
\frac{\mathrm{d} R}{\mathrm{~d} t}=-k R . \quad(k=\mathrm{constant})
$$

This is first order and linear.
Example 5.4 (Simple pendulum).

$$
\frac{\mathrm{d}^{2} \theta}{\mathrm{~d} t^{2}}+\frac{g}{l} \sin \theta=0
$$

It is a second-order ODE. However it is non-linear, due to the $\sin \theta$ term.


Figure 5.2: An simple pendulum comprised of an object with mass $m$ attached to a string with length $l$. The other end of the string is attached to a ceiling.

Partial differential equations (PDEs) involve partial derivatives (see Chapter 3), such as...

Example 5.5 (Beam Equation). The Beam Equation provides a model for the load carrying and deflection properties of beams, and is given by

$$
\frac{\partial^{2} u}{\partial t^{2}}+c^{2} \frac{\partial^{4} u}{\partial x^{4}}=0 .
$$

... but you won't see them in this course. You'll have to wait until Maths for Engineers 3 (MATH6503) for that!

### 5.2 First order separable ODEs

An ODE $\frac{\mathrm{d} y}{\mathrm{~d} x}=F(x, y)$ is separable if we can write $F(x, y)=f(x) g(y)$ for some functions $f(x), g(y)$.

## Example 5.6.

$$
\begin{aligned}
& \frac{\mathrm{d} y}{\mathrm{~d} x}=y \quad \text { IS separable } \\
& \frac{\mathrm{d} y}{\mathrm{~d} x}=x^{2}-y^{2} \quad \text { IS NOT. }
\end{aligned}
$$

Example 5.7. Find the general solution to the ODE

$$
9 y \frac{\mathrm{~d} y}{\mathrm{~d} x}+4 x=0 .
$$

"Separating the variables", we have

$$
\begin{aligned}
9 y \mathrm{~d} y & =-4 x \mathrm{~d} x \Longleftrightarrow \\
9 \int y \mathrm{~d} y & =-4 \int x \mathrm{~d} x \\
\frac{9}{2} y^{2} & =-\frac{4}{2} x^{2}+C,
\end{aligned}
$$

i.e. the general solution is

$$
\frac{x^{2}}{9}+\frac{y^{2}}{4}=K, \quad(K=C / 36)
$$

which describes a 'family' of ellipses.
We can check our solution by differentiating:

$$
\frac{2}{9} x+\frac{2}{4} y y^{\prime}=0
$$

i.e

$$
9 y y^{\prime}+4 x=0 .
$$

Example 5.8. Find the general solution to

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=\frac{y+1}{x+1} .
$$

$$
\begin{aligned}
& \Rightarrow \quad \int \frac{1}{y+1} \mathrm{~d} y=\int \frac{1}{x+1} \mathrm{~d} x \\
& \Rightarrow \quad \ln |y+1|=\ln |x+1|+C .
\end{aligned}
$$

Use $\log \left(\frac{a}{b}\right)=\log a-\log b$ :

$$
\ln \left|\frac{y+1}{x+1}\right|=C,
$$

or

$$
\frac{y+1}{x+1}=e^{C}=K .
$$

Again we can easily check this using differentiation.
Example 5.9. Solve the ODE

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}=1+y^{2}
$$

Separating variables:

$$
\begin{array}{rlrl} 
& & \int \frac{\mathrm{d} y}{1+y^{2}} & =\int \mathrm{d} x \\
\Rightarrow & \arctan y & =x+C \\
& \Rightarrow & y & =\tan (x+C) .
\end{array}
$$

Once again, this is easily checked by differentiation.
Example 5.10 (2007 Exam Question). Solve

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}-\frac{y(y+1)}{x(x-1)}=0
$$

finding $y$ explicitly, i.e $y=f(x)$.
Solution: This equation is separable, thus separating the variables and integrating gives

$$
\begin{aligned}
\frac{\mathrm{d} y}{\mathrm{~d} x} & =\frac{y(y+1)}{x(x-1)} \\
\int \frac{\mathrm{d} y}{y(y+1)} & =\int \frac{\mathrm{d} x}{x(x-1)} .
\end{aligned}
$$

To solve the integrals, use partial fractions:

$$
\begin{aligned}
\int\left[\frac{1}{y}-\frac{1}{y+1}\right] \mathrm{d} y & =\int\left[-\frac{1}{x}+\frac{1}{x-1}\right] \mathrm{d} x \\
\ln y-\ln (y+1) & =-\ln x+\ln (x-1)+C \\
\ln \left(\frac{y}{y+1}\right) & =\ln \left(\frac{x-1}{x}\right)+C \\
\frac{y+1}{y} & =e^{-C} \frac{x}{x-1} .
\end{aligned}
$$

Let $K=e^{C}$. Then

$$
\begin{aligned}
y & =(y+1)\left(\frac{x-1}{K x}\right) \\
y\left[1-\left(\frac{x-1}{K x}\right)\right] & =\left(\frac{x-1}{K x}\right) \\
y(K x-x+1) & =x-1 .
\end{aligned}
$$

$$
\therefore \quad y=\frac{x-1}{K x-x+1}
$$

is the explicit solution.
Example 5.11 (2010 Exam Question). Solve

$$
\left(y+x^{2} y\right) \frac{\mathrm{d} y}{\mathrm{~d} x}=1
$$

Solution:

$$
\begin{array}{r}
y\left(1+x^{2}\right) \frac{\mathrm{d} y}{\mathrm{~d} x}=1 \\
\int y \mathrm{~d} y=\int \frac{\mathrm{d} x}{x^{2}+1} \\
\frac{y^{2}}{2}=\arctan x+C
\end{array}
$$

i.e. the solution is $y= \pm \sqrt{2 \arctan x+2 C}$.

### 5.3 First order linear ODEs

Aside: Exact types An exact type is where the LHS of the differential equation is the exact derivative of the product.

Example 5.12.

$$
\begin{aligned}
& x \frac{\mathrm{~d} y}{\mathrm{~d} x}+y=e^{x} \\
\Rightarrow \quad & \frac{\mathrm{~d}}{\mathrm{~d} x}(x y)=e^{x} \\
\Rightarrow \quad & x y=e^{x}+C .
\end{aligned}
$$

## Example 5.13.

$$
\begin{aligned}
& e^{x} e^{y} \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{x} e^{y}=e^{2 x} \\
\Rightarrow \quad & \frac{\mathrm{~d}}{\mathrm{~d} x}\left(e^{x} e^{y}\right)=e^{2 x} \\
\Rightarrow & e^{x} e^{y}=\frac{1}{2} e^{2 x}+C .
\end{aligned}
$$

I recommend that you bear this in mind as we proceed...
First order linear ODEs are equations that may be written in the form:

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} x}+P(x) y=Q(x) . \tag{5.2}
\end{equation*}
$$

## Example 5.14.

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}+y \cot x=\operatorname{cosec} x . \quad[P(x)=\cot x, Q(x)=\operatorname{cosec} x]
$$

## Example 5.15.

$$
\begin{aligned}
& \tan x \frac{\mathrm{~d} y}{\mathrm{~d} x}+y \\
=\quad & e^{x} \tan x \\
\Rightarrow \quad \frac{\mathrm{~d} y}{\mathrm{~d} x}+\cot x y & =e^{x} . \quad\left[P(x)=\cot x, Q(x)=e^{x}\right]
\end{aligned}
$$

In general, Equation (5.2) is NOT exact.
Big question: Can we multiply the equation by a function of $x$ which will make it exact?

Let's suppose we can, and call this function $I(x)$; the integrating factor (IF). Then multiply both sides of (5.2) by $I$ :

$$
\underbrace{I \frac{\mathrm{~d} y}{\mathrm{~d} x}+I P y}_{\text {Exact type }}=I Q .
$$

Compare the LHS with

$$
\overbrace{I \frac{\mathrm{~d} y}{\mathrm{~d} x}+\frac{\mathrm{d} I}{\mathrm{~d} x} y}^{\frac{\mathrm{d}}{x}(I y)}
$$

Hence we require

$$
\begin{aligned}
& I P y=\frac{\mathrm{d} I}{\mathrm{~d} x} y \\
\Rightarrow & \frac{\mathrm{~d} I}{\mathrm{~d} x}=I P \\
\Rightarrow & \int \frac{\mathrm{~d} I}{I}=\int P \mathrm{~d} x \\
\Rightarrow & \ln I=\int P \mathrm{~d} x \quad \text { [No need for integration constants!] } \\
\Rightarrow & \ln I=e^{\int P \mathrm{~d} x},
\end{aligned}
$$

and this is the IF. We will substitute this into (5.2):

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}+P(x) y=Q(x)
$$

Multiply by $I$ :

$$
\begin{aligned}
& e^{\int P \mathrm{~d} x} \frac{\mathrm{~d} y}{\mathrm{~d} x}+e^{\int P \mathrm{~d} x} P y=e^{\int P \mathrm{~d} x} Q \\
\Rightarrow & \frac{\mathrm{~d}}{\mathrm{~d} x}\left(y e^{\int P \mathrm{~d} x}\right)=e^{\int P \mathrm{~d} x} Q \\
\Rightarrow & y I=\int e^{\int P \mathrm{~d} x} Q \mathrm{~d} x
\end{aligned}
$$

This is the form we end up with.
I will not ask you to go through this derivation in the exam. However, you will need to know how to apply it.

Example 5.16. Solve

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}+2 y=e^{-x}
$$

We require the IF:

$$
I=e^{\int P \mathrm{~d} x}=e^{\int 2 \mathrm{~d} x}=e^{2 x}
$$

Then

$$
\begin{aligned}
& e^{2 x} \frac{\mathrm{~d} y}{\mathrm{~d} x}+2 e^{2 x} y=e^{2 x} e^{-x} \\
\Rightarrow \quad & \frac{\mathrm{~d}}{\mathrm{~d} x}\left(y e^{2 x}\right)=e^{x} \\
\Rightarrow & y e^{2 x}=e^{x}+C
\end{aligned}
$$

or

$$
y=e^{-x}+C e^{-2 x}
$$

Example 5.17. Solve

$$
\cos x \frac{\mathrm{~d} y}{\mathrm{~d} x}+y \sin x=\frac{1}{2} \sin 2 x
$$

Get it into the right form first!

$$
\begin{align*}
& \Rightarrow \frac{\mathrm{d} y}{\mathrm{~d} x}+y \tan x=\frac{\sin 2 x}{2 \cos x}=\frac{2 \sin x \cos x}{22 \cos x} \\
& \Rightarrow \frac{\mathrm{~d} y}{\mathrm{~d} x}+y \tan x=\sin x \tag{5.3}
\end{align*}
$$

so $P(x)=\tan x$. Now seek the IF:

$$
I=e^{\int P \mathrm{~d} x}=e^{\int \tan x \mathrm{~d} x}=e^{-\ln (\cos x)}=\frac{1}{e^{\ln (\cos x)}}=\frac{1}{\cos x}
$$

A VERY common error: $e^{-\ln (\cos x)}=\cos x$.

Multiply (5.3) throughout by I to give

$$
\frac{1}{\cos x} \frac{\mathrm{~d} y}{\mathrm{~d} x}+\frac{\tan x}{\cos x} y=\tan x
$$

i.e.

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left(\frac{y}{\cos x}\right)=\tan x \\
\Rightarrow & \frac{y}{\cos x}=\int \tan x \mathrm{~d} x+C=-\ln (\cos x)+C
\end{aligned}
$$

Therefore the general solution is

$$
y=C \cos x-\cos x \ln (\cos x)
$$

Example 5.18. Solve

$$
x \frac{\mathrm{~d} y}{\mathrm{~d} x}+=x^{2}+3 y
$$

Get it in the right form first. . .

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} x}-\frac{3}{x} y=x \tag{5.4}
\end{equation*}
$$

Find the integrating factor

$$
I(x)=e^{\int-\frac{3}{x} \mathrm{~d} x}=e^{-3 \ln x}=e^{\ln \left(x^{-3}\right)}=x^{-3}
$$

Now multiply both sides of (5.4) by the integrating factor to make the LHS an exact type:

$$
x^{-3} \frac{\mathrm{~d} y}{\mathrm{~d} x}-3 x^{-4} y=x^{-2} \frac{\partial}{\partial x}\left(x^{-3} y\right) \quad=x^{-2}
$$

and integrate both sides of the equation to gain

$$
\begin{aligned}
x^{-3} y & =-x^{-1}+C \\
y & =x^{3}\left(C-x^{-1}\right) \\
y=x^{2}(C x-1) &
\end{aligned}
$$

### 5.4 Initial Value Problems

All the solutions we obtained so far contain an annoying constant of integration $C$. When engineers work with ODEs, they are interested in a particular solution satisfying the given initial condition.

An ODE together with an initial condition (IC) is called an initial value problem (IVP). In other words:

$$
\mathrm{ODE}+\mathrm{IC}=\mathrm{IVP}
$$

We need only two steps to solve an IVP:

1 ODE: Find the general solution, containing an arbitrary constant.
2 IC: Apply the condition to determine the arbitrary constant. Usually, the condition is given as

$$
y\left(x_{0}\right)=y_{0}
$$

which tells us that when $x=x_{0}, y=y_{0}$.
Example 5.19. Solve the IVP

$$
2 \frac{\mathrm{~d} y}{\mathrm{~d} x}-4 x y=2 x, \quad y(0)=0
$$

Start by rewriting in the form

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}-2 x y=x
$$

which is a first order linear equation, so we calculate the IF:

$$
\begin{gathered}
I=e^{\int-2 x \mathrm{~d} x}=e^{-x^{2}} \\
\therefore \quad \frac{\mathrm{~d} y}{\mathrm{~d} x} e^{-x^{2}}-2 x e^{-x^{2}} y=x e^{-x^{2}}
\end{gathered}
$$

Hence

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{~d} x}\left(y e^{-x^{2}}\right)=x e^{-x^{2}} \\
\Rightarrow & y e^{-x^{2}}=\int x e^{-x^{2}} \mathrm{~d} x \\
\Rightarrow & y e^{-x^{2}}=-\frac{1}{2} e^{-x^{2}}+C \\
\Rightarrow & y=-\frac{1}{2}+C e^{x^{2}}
\end{aligned}
$$

Now apply the IC $y(0)=0$. This gives

$$
0=-\frac{1}{2}+C \quad \Rightarrow C=\frac{1}{2}
$$

and so the solution is

$$
y=\frac{1}{2}\left(e^{x^{2}}-1\right)
$$

Example 5.20. Solve the IVP

$$
x \frac{\mathrm{~d} y}{\mathrm{~d} x}+2 y=4 x^{2}, \quad y(1)=2
$$

Get the equation in the right form first!

$$
\frac{\mathrm{d} y}{\mathrm{~d} x}+\frac{2}{x} y=4 x
$$

Then the IF is:

$$
\begin{aligned}
& I=\mathrm{e}^{\int \frac{2}{x} \mathrm{~d} x}=e^{2 \ln x}=e^{\ln x^{2}}=x^{2} . \\
\Rightarrow & x^{2} \frac{\mathrm{~d} y}{\mathrm{~d} x}+2 x y=4 x^{3} \\
\Rightarrow & \frac{\mathrm{~d}}{\mathrm{~d} x}\left(x^{2} y\right)=4 x^{3} \\
\Rightarrow & x^{2} y=x^{4}+C \\
\Rightarrow & y=x^{2}+C x^{-2} .
\end{aligned}
$$

Apply the condition $y(1)=2$ :

$$
y(1)=1+C=2 \Rightarrow C=1 .
$$

So the solution is

$$
y=x^{2}+\frac{1}{x^{2}}
$$

Example 5.21 (Logistic Equation). Suppose the rate of change of $x$ is proportional to:

$$
r x(1-x)
$$

where $r>0$ is constant. Show that if initially $x=x_{0}($ at $t=0)$ and $0<x_{0}<1$, then $\lim _{t \rightarrow \infty} x=1$.

First, we set up the ODE:

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=r x(1-x)
$$

which is the logistic equation. This ODE has applications in many fields of study such as ecology, psychology, chemistry and even politics!

The logistic equation can be tackled by separating variables...

$$
\begin{aligned}
\int \frac{\mathrm{d} x}{x(1-x)} & =r \int \mathrm{~d} t \\
\int\left[\frac{1}{x}+\frac{1}{1-x}\right] \mathrm{d} x & =r t+C \\
\ln |x|-\ln |1-x| & =r t+C \\
\ln \left|\frac{x}{1-x}\right| & =r t+C \\
\frac{x}{1-x} & =e^{r t+C}=e^{r t} e^{C}
\end{aligned}
$$

and let $G=e^{C}$. We then make $x$ the subject...

$$
\begin{aligned}
x & =(1-x) G e^{r t} \\
x & =G e^{r t}-x G e^{r t} \\
x\left(1+G e^{r t}\right) & =G e^{r t}
\end{aligned}
$$

which leads to

$$
x=\frac{G e^{r t}}{1+G e^{r t}}
$$

Next, find $G$ using the initial condition:

$$
x_{0}=\frac{1}{\frac{1}{G}+1}, \quad \Rightarrow \quad \frac{1}{G}=\frac{1}{x_{0}}-1
$$

and therefore

$$
x(t)=\frac{1}{1+\left(\frac{1}{x_{0}}-1\right)} e^{-r t}=\frac{x_{0}}{x_{0}+\left(1-x_{0}\right) e^{-r t}}
$$

the so-called logistic function. Finally, we note that as $t \rightarrow \infty, x(t) \rightarrow \frac{x / 0}{x / 5}=1$, as intended.


Figure 5.3: A plot depicting the logistic curve. Here, $x_{0}=0.01$ and $r=0.2$.

## Chapter 6

## Vectors

### 6.1 Introduction

Definition 6.1. A vector is a quantity with both a magnitude (size) and direction.

Many quantities in engineering applications can be described by vectors, e.g. force, velocity, magnetic field.

They can be represented by arrows, for example...


Figure 6.1: Some vectors.

Magnitude=Length of $A B$
Direction is shown in the Figure 6.1.
We will write $\overrightarrow{A B}$ or a to represent the top vector in the figure.
Two vectors are equal when they have both the same magnitude and direction. So $\overrightarrow{A B}=\overrightarrow{C D}$.

But $\overrightarrow{A B} \neq \overrightarrow{E F}$, since both the magnitude and direction are different.
The sum of two vectors $\mathbf{a}$ and $\mathbf{b}$ is found by adding the vectors "head to tail":


Example 6.1 (Forces on an object). Consider the following forces acting on an object:


Forces add to give a net effect or resultant force.

$$
\begin{aligned}
& \qquad \mathbf{R}=\mathbf{F}_{\mathbf{1}}+\mathbf{F}_{\mathbf{2}} \\
& \text { Magnitude: }|\mathbf{R}|=\sqrt{8^{2}+5^{2}} \approx 9.4 \mathrm{~N} \\
& \text { Direction: Use } \tan \theta=\frac{\left|\mathbf{F}_{\mathbf{1}}\right|}{\left|\mathbf{F}_{\mathbf{2}}\right|}=\frac{8}{5}=1.6 \\
& \Rightarrow \theta=58^{\circ}
\end{aligned}
$$

You can multiply a vector a by a scalar (number) $k$. Then, as shown in Figure 6.2, if $k>0$, $k \mathbf{a}$ is a vector in the same direction as $\mathbf{a}$, and the magnitude is $k|\mathbf{a}| \ldots$ BUT if $k<0, k \mathbf{a}$ is in the opposite direction!

Example 6.2. Two points $A$ and $B$ have position vectors (i.e. relative to a fixed origin $O) \mathbf{a}$ and $\mathbf{b}$ respectively. What is the position vector of a point on the line joining $A$ and $B$, equidistant from $A$ and $B$ ?

Well, the first thing we need is a sketch of the problem, like in Figure 6.3.
Next, note that $\overrightarrow{A B}=\mathbf{b}-\mathbf{a}$.


Figure 6.2: Two examples of scalar multiplication of the vector $\mathbf{a}$.


Figure 6.3: In this sketch, $X$ is the midpoint of the line joining $A$ and $B$

$$
\begin{aligned}
\mathbf{x} & =\mathbf{a}+\overrightarrow{A X}=\mathbf{a}+\frac{1}{2} \overrightarrow{A B} \\
& =\mathbf{a}+\frac{1}{2}(\mathbf{b}-\mathbf{a}) \\
& =\frac{1}{2}(\mathbf{a}+\mathbf{b})
\end{aligned}
$$

Definition 6.2. A unit vector is a vector with magnitude 1.

Often represented using a hat symbol:

For any vector $\mathbf{a}$,

$$
\begin{gathered}
\hat{\mathbf{a}}=\frac{\mathbf{a}}{|\mathbf{a}|} \quad \text { is a unit vector since } \\
\qquad|\mathbf{a}|=\left|\frac{\mathbf{a}}{|\mathbf{a}|}\right|=\frac{|\mathbf{a}|}{|\mathbf{a}|}=1
\end{gathered}
$$

Unit vectors in the $x, y, z$ idrections are denoted $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively.
Then the position of a point $P$ from the origin, with coordinates $(x, y, z)$, is

$$
\mathbf{r}=x \mathbf{i}+y \mathbf{j}+z \mathbf{k}
$$



Figure 6.4: ijk

## Example 6.3.

$$
\begin{aligned}
& \mathbf{a}=6 \mathbf{i}-3 \mathbf{j}+\mathbf{k}, \\
& \mathbf{b}=4 \mathbf{i}+2 \mathbf{j} .
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{a}+\mathbf{b} & =10 \mathbf{i}-\mathbf{j}-\mathbf{k} \\
\mathbf{b}-\mathbf{a} & =-2 \mathbf{i}+5 \mathbf{j}-\mathbf{k} \\
3 \mathbf{a} & =18 \mathbf{i}-9 \mathbf{j}+3 \mathbf{k} .
\end{aligned}
$$

For a position vector $\mathbf{r}=x \mathbf{i}+y \mathbf{j}+z \mathbf{k}$, the magnitude is

$$
|\mathbf{r}|=\sqrt{x^{2}+y^{2}+z^{2}} .
$$

Then for the previous example,

$$
\begin{aligned}
& |\mathbf{a}|=\sqrt{6^{2}+(-3)^{2}+1^{2}}=\sqrt{46} \\
& |\mathbf{b}|=\sqrt{4^{2}+2^{2}+0^{2}}=2 \sqrt{5}
\end{aligned}
$$

So far we've seen how to add two vectors. Now we have a question...
Q: How can we multiply two vectors together?
I'm going to show you that there are in fact two ways to multiply vectors...

### 6.2 The Dot Product

Let us consider the origin of the dot product:
We take two vectors $\mathbf{a}$ and $\mathbf{b}$ :

We might be interested in the length of the component of a which is in the same direction as $\mathbf{b}$.

Here $0 \leq \theta<\pi$ is the angle between $\mathbf{a}$ and $\mathbf{b}$.


Figure 6.5: The two vectors $\mathbf{a}$ and $\mathbf{b}$. We see that the length of the component of a which is in the same direction as $\mathbf{b}$ is $|\mathbf{a}| \cos \theta$.

Compare with the dot product formula:

$$
\mathbf{a} \cdot \mathbf{b}=|\mathbf{a}||\mathbf{b}| \cos \theta
$$

Looks almost like the length of the component of $\mathbf{a}$, but is rescaled such that we have the symmetry:

$$
\mathbf{a} \cdot \mathbf{b}=\mathbf{b} \cdot \mathbf{a}
$$

So the dot product also gives us a rescaling of the length of the component of $\mathbf{b}$ in the same direction as a. But we expected that in the first place, because of the above symmetry rule!


Figure 6.6: This time, we would like the length of the component of $\mathbf{b}$ which is in the same direction as $\mathbf{a}$. That length is $|\mathbf{b}| \cos \theta$.

Note that

$$
\mathbf{a} \cdot \mathbf{b}=|\mathbf{a}||\mathbf{b}| \cos \theta \quad \Rightarrow \quad \cos \theta=\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} ;
$$

which is a useful method for calculating $\theta$ if you know $\mathbf{a}$ and $\mathbf{b}$.
Two non-zero vectors are perpendicular (orthogonal) if and only if their dot product is zero, i.e.

$$
\begin{aligned}
\mathbf{a} \cdot \mathbf{b}=0 & \Rightarrow \quad|\mathbf{a}||\mathbf{b}| \cos \theta=0 \\
& \Rightarrow \cos \theta=0 \\
& \Rightarrow \theta=\frac{\pi}{2} \quad\left(90^{\circ}\right)
\end{aligned}
$$

Now consider $\mathbf{i}, \mathbf{j}, \mathbf{k}$. These are unit vectors, and are mutually perpendicular. These two facts combined show that, e.g.

$$
\mathbf{i} \cdot \mathbf{i}=1, \quad \mathbf{i} \cdot \mathbf{j}=0, \quad \text { etc. }
$$

so if you then let

$$
\begin{array}{ll}
\mathbf{a}=\left(a_{1}, a_{2}, a_{3}\right) & \left(=a_{1} \mathbf{i}+a_{2} \mathbf{j}+a_{3} \mathbf{k}\right) \\
\mathbf{b}=\left(b_{1}, b_{2}, b_{3}\right) & \left(=b_{1} \mathbf{i}+b_{2} \mathbf{j}+b_{3} \mathbf{k}\right)
\end{array}
$$

and multiply out $\mathbf{a} \cdot \mathbf{b}$, you obtain

$$
\mathbf{a} \cdot \mathbf{b}=a_{1} b_{1}+a_{2} b_{2}+a_{3} b_{3}
$$

## Note:

$$
\begin{aligned}
& \mathbf{a} \cdot \mathbf{a}=|\mathbf{a}||\mathbf{a}| \cos 0=|\mathbf{a}|^{2} \\
& \text { i.e. } \quad|\mathbf{a}|=\sqrt{\mathbf{a} \cdot \mathbf{a}} .
\end{aligned}
$$

Let's try this with $\mathbf{r}=x \mathbf{i}+y \mathbf{j}+x \mathbf{k}$. Then:

$$
|\mathbf{r}|=\sqrt{\mathbf{r} \cdot \mathbf{r}}=\sqrt{x^{2}+y^{2}+z^{2}}
$$

which is consistent with the earlier formula for the magnitude of $\mathbf{r}$.
Example 6.4. For

$$
\begin{aligned}
& \mathbf{a}=6 \mathbf{i}-3 \mathbf{j}+\mathbf{k} \\
& \mathbf{b}=4 \mathbf{i}+2 \mathbf{j}
\end{aligned}
$$

calculate $\mathbf{a} \cdot \mathbf{b}$ and find the angle between the two vectors.

$$
\mathbf{a} . \mathbf{b}=6 \times 4+(-3) \times 2+1 \times(0)=18
$$

But recall

$$
\mathbf{a} \cdot \mathbf{b}=|\mathbf{a}||\mathbf{b}| \cos \theta
$$

and that

$$
|\mathbf{a}|=\sqrt{46}, \quad|\mathbf{b}|=2 \sqrt{5},
$$

therefore

$$
\begin{gathered}
\cos \theta=\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{18}{2 \sqrt{5} \sqrt{46}}=0.593 . \\
\therefore \quad \theta=\cos ^{-1}(0.593)=53.6^{\circ}
\end{gathered}
$$

Example 6.5. Points $A, B$ and $C$ have coordinates $(3,2),(4,-3),(7,-5)$ respectively.
i Find $\overrightarrow{A B}$ and $\overrightarrow{A C}$.
ii Find $\overrightarrow{A B} \cdot \overrightarrow{A C}$.
iii Deduce the angle between $\overrightarrow{A B}$ and $\overrightarrow{A C}$.
i

$$
\begin{aligned}
& \overrightarrow{A B}=(4 \mathbf{i}-3 \mathbf{j})-(3 \mathbf{i}+2 \mathbf{j})=\mathbf{i}-5 \mathbf{j} \\
& \overrightarrow{A C}=(7 \mathbf{i}-5 \mathbf{j})-(3 \mathbf{i}+2 \mathbf{j})=4 \mathbf{i}-7 \mathbf{j}
\end{aligned}
$$

ii Now for the dot product:

$$
\overrightarrow{A B} \cdot \overrightarrow{A C}=4 \times 1+(-5) \times(-7)=4+35=39
$$

iii To calculate the angle, note that

$$
\begin{aligned}
& |\overrightarrow{A B}|=\sqrt{1^{2}+(-5)^{2}}=\sqrt{26} \\
& |\overrightarrow{A C}|=\sqrt{4^{2}+(-7)^{2}}=\sqrt{65}
\end{aligned}
$$

Then

$$
\cos \theta=\frac{\overrightarrow{A B} \cdot \overrightarrow{A C}}{|\overrightarrow{A B}||\overrightarrow{A C}|}=\frac{39}{\sqrt{26} \sqrt{65}}=0.949 \quad \text { (3 d.p.) }
$$

which gives $\theta=18^{\circ}$.

So far, we have seen one way to multiply two vectors together. However, that first way, the dot product, spits out a number. It would be nice if there was a way to multiply two vectors together such that the result is another vector (Guess what? There is one!)

### 6.3 The Cross Product

Take any two vectors $\mathbf{a}$ and $\mathbf{b}$. Then the cross product is denoted as

$$
\mathbf{a} \times \mathbf{b}
$$

Before giving the definition, let's consider the motivation behind it using a physics context...
Example 6.6 (Moments). Consider a seesaw. If I apply a force on it at some point away from the pivot, it will turn. Also, if the force is applied farther away from the pivot, the seesaw will turn more easily.


$$
\begin{aligned}
\mathbf{r} & =\text { Position where the force is exerted } \\
\mathbf{F} & =\text { The force applied, }
\end{aligned}
$$

then the moment of $\mathbf{F}$ about a point $O$ is

$$
m=|\mathbf{F}| d
$$

where

$$
d=|\mathbf{r}| \sin \theta
$$

is the perpendicular distance between $O$ and the line of action of $\mathbf{F}$.

$$
\therefore \quad m=|\mathbf{r}||\mathbf{F}| \sin \theta
$$

In fact, the moment vector of $\mathbf{F}$ about $O$, i.e. $\mathbf{m}$, is

$$
\mathbf{m}=\mathbf{r} \times \mathbf{F}
$$

which is perpendicular to both $\mathbf{r}$ and $\mathbf{F}$. Moreover, $\mathbf{m}$ points in the same direction as the axis of rotation for the seesaw (here, $\mathbf{m}$ points out of the page).

Now, $m=|\mathbf{m}|$, hence the magnitude of $\mathbf{m}$ is:

$$
|\mathbf{m}|=|\mathbf{r}||\mathbf{F}| \sin \theta
$$

Okay, now I can define the vector product:
Definition 6.3. The cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$
\mathbf{a} \times \mathbf{b}=|\mathbf{a}||\mathbf{b}| \sin \theta \hat{\mathbf{n}},
$$

which is a VECTOR, not a NUMBER. So try not to confuse this with the dot product.

$$
\begin{aligned}
\text { Length of } \mathbf{a} \times \mathbf{b}: & |\mathbf{a} \times \mathbf{b}|=|\mathbf{a}||\mathbf{b}| \sin \theta . \\
\text { Direction of } \mathbf{a} \times \mathbf{b}: & \hat{\mathbf{n}}, \text { found using the right hand rule. }
\end{aligned}
$$

$\hat{\mathbf{n}}$ is a unit vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$.


Figure 6.7: The vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. If you put your thumb on $\mathbf{a}$ and your index finger on $\mathbf{b}$, then your middle finger will tell you the direction of $\mathbf{a} \times \mathbf{b}$.

This definition only works for 3D vectors!

Q: Now, does $\mathbf{a} \times \mathbf{b}=\mathbf{b} \times \mathbf{a}$ ?

A: NO!
To see this, let $\mathbf{v}=\mathbf{a} \times \mathbf{b}$ and $\mathbf{w}=\mathbf{b} \times \mathbf{a}$. By definition, we will have that $|\mathbf{v}|=|v|$, but what about their directions? Well, the right hand rule shows us that $\mathbf{v}=-\mathbf{w}$. Hence

$$
\mathbf{b} \times \mathbf{a} \neq \mathbf{a} \times \mathbf{b}!
$$

Suppose we have any two vectors a and b. If:

$$
\begin{array}{ll}
\mathbf{a}=a_{1} \mathbf{i}+a_{2} \mathbf{j}+a_{3} \mathbf{k} & =\left(a_{1}, a_{2}, a_{3}\right) \\
\mathbf{b}=b_{1} \mathbf{i}+b_{2} \mathbf{j}+b_{3} \mathbf{k} & =\left(b_{1}, b_{2}, b_{3}\right),
\end{array}
$$

then the three components of $\mathbf{a} \times \mathbf{b}$ are:

$$
\mathbf{a} \times \mathbf{b}=\left(a_{2} b_{3}-a_{3} b_{2}\right) \mathbf{i}+\left(a_{3} b_{1}-a_{1} b_{3}\right) \mathbf{j}+\left(a_{1} b_{2}-a_{2} b_{1}\right) \mathbf{k}
$$

This can be conveniently represented using a $3 \times 3$ matrix determinant:

$$
\mathbf{a} \times \mathbf{b}=\left\lvert\, \begin{array}{ccc|cc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
a_{1} & a_{2} & a_{3} & a_{1} & a_{2} \\
b_{1} & b_{2} & b_{3} & b_{1} & b_{2}
\end{array}\right.
$$

A trick to calculate the determinant is to multiply along each of the six diagonal lines. Next, add all the products corresponding to the green diagonals, and then subtract all the products for the red diagonals. In other words,

Determinant $=$ Sum of the green products - Sum of red products.
Example 6.7. Compute $\mathbf{a} \times \mathbf{b}$, where

$$
\begin{gathered}
\mathbf{a}=4 \mathbf{i}-\mathbf{k} \\
\mathbf{b}=-2 \mathbf{i}+\mathbf{j}+3 \mathbf{k} \\
\mathbf{a} \times \mathbf{b}=\left\lvert\, \begin{array}{ccc|cc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
4 & 0 & -1 & 4 & 0 \\
-2 & 1 & 3 & -2 & 1
\end{array}\right. \\
=0 \mathbf{i}+2 \mathbf{j}+4 \mathbf{k}-0 \mathbf{k}-(-\mathbf{i})-12 \mathbf{j} \\
=\mathbf{i}-10 \mathbf{j}+4 \mathbf{k} .
\end{gathered}
$$

Example 6.8. Show that $\mathbf{i} \times \mathbf{j}=\mathbf{k}$.

$$
\begin{aligned}
& \mathbf{i} \times \mathbf{j}=\left\lvert\, \begin{array}{ccc|cc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{array}\right. \\
&=0 \mathbf{i}+0 \mathbf{j}+1 \mathbf{k}-0 \mathbf{i}-0 \mathbf{j}-0 \mathbf{k} \\
&= \mathbf{k}
\end{aligned}
$$

Remark 6.1. A nice interpretation of the length $|\mathbf{a} \times \mathbf{b}|$ is that if $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$, then this is the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$, i.e.

$$
A=\underbrace{|\mathbf{a}|}_{\text {Base length }} \underbrace{|\mathbf{b}| \sin \theta}_{\text {Height }}
$$



Figure 6.8: A paralellogram, whose sides correspond to vectors $\mathbf{a}$ and $\mathbf{b}$. It can be split into two triangles.

Proof:

$$
A=2 A_{1}
$$

but

$$
\begin{aligned}
A_{1} & =\frac{1}{2}|\mathbf{a}||\mathbf{b}| \sin \theta, \quad \text { [Anyone recognise this trigonometric formula?] } \\
& =\frac{1}{2}|\mathbf{a} \times \mathbf{b}|
\end{aligned}
$$

hence

$$
A=|\mathbf{a} \times \mathbf{b}| .
$$

Example 6.9 (Recycled exam question!). Find the area of a triangle with adjacent sides given by

$$
\begin{aligned}
& \mathbf{a}=\mathbf{i}+2 \mathbf{j}-\mathbf{k} \\
& \mathbf{b}=\mathbf{j}+\mathbf{k}
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \mathbf{i} \times \mathbf{j}=\left\lvert\, \begin{array}{ccc|cc}
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\
1 & 2 & -1 & 1 & 2 \\
0 & 1 & 1 & 0 & 1
\end{array}\right. \\
& =2 \mathbf{i}+0 \mathbf{j}+\mathbf{k}-(-\mathbf{i})-\mathbf{j}-0 \mathbf{k} \\
& =3 \mathbf{i}-\mathbf{j}+\mathbf{k} .
\end{aligned}
$$

We want the area of the shaded region $A$, but

$$
\begin{aligned}
|\mathbf{a} \times \mathbf{b}| & =2 A \\
\Rightarrow A & =\frac{1}{2}|\mathbf{a} \times \mathbf{b}| \\
& =\frac{1}{2} \sqrt{3^{2}+(-1)^{2}+1^{2}} \\
& =\frac{1}{2} \sqrt{11}
\end{aligned}
$$

## Chapter 7

## Numerical Methods

### 7.1 Introduction

In many cases the integral

$$
\mathscr{I}=\int_{a}^{b} f(x) \mathrm{d} x
$$

can be found by finding a function $F(x)$ such that $F^{\prime}(x)=f(x)$, and using

$$
\mathscr{I}=\int_{a}^{b} f(x) \mathrm{d} x=F(b)-F(a)
$$

which is known as the analytical (exact) solution.
But consider

$$
\int_{0}^{1} \sqrt{1+x^{3}} \mathrm{~d} x, \quad \text { and } \quad \int_{0}^{1} e^{x^{2}} \mathrm{~d} x
$$

Neither of the above integrals can be expressed in terms of functions that we know. However both of these integrals do exist, since they both represent the area below the curves $\sqrt{1+x^{3}}$ and $e^{x^{2}}$ between $x=0$ and $x=1$ (and both curves are well-behaved).

Yet in the above two cases we know what $f(x)$ is. Sometimes, engineers want to calculate an area by computing $I$, but...

- They don't know the equation for $f(x)$.
- There might be no formula for $f(x)$ at all!

Thankfully, there are some practical methods out there for calculating areas under graphs, e.g. counting squares. But this is time-consuming and boring! Besides, there are other methods of calculating areas which are much more accurate, even though they are still only approximations.

### 7.2 The Rectangular Rule

The rectangular rule (also called the midpoint rule) is perhaps the simplest of the three methods for estimating an integral you will see in the course.


Figure 7.1: The main idea of the Rectangular Rule is that we can approximate the area unfer a curve $y=f(x)$ by lots of small rectangles, each with width $h$.

- Integrate over an interval $a \leq x \leq b$.
- Divide this interval up into $n$ equal subintervals of length $h=(b-a) / n$.
- Approximate $f$ in each subinterval by $f\left(x_{j}^{*}\right)$, where $x_{j}^{*}$ is the midpoint of the subinterval.
- Area of each rectangle: $f\left(x_{1}^{*}\right) h, f\left(x_{2}^{*}\right) h, \ldots, f\left(x_{n}^{*}\right) h$.

$$
\therefore \quad \mathscr{I}=\int_{a}^{b} f(x) \mathrm{d} x \approx h\left[f\left(x_{1}^{*}\right)+f\left(x_{2}^{*}\right)+\cdots+f\left(x_{n}^{*}\right)\right] .
$$

The approximation on the RHS becomes more accurate as more rectangles are used. In fact,

$$
\int_{a}^{b} f(x) \mathrm{d} x=\lim _{h \rightarrow 0}\left\{h\left[f\left(x_{1}^{*}\right)+f\left(x_{2}^{*}\right)+\cdots+f\left(x_{n}^{*}\right)\right]\right\}
$$

Note: As $h \rightarrow 0, n \rightarrow \infty$, since $h=\frac{b-a}{n}$ and $(b-a)$ is fixed.
Remark 7.1. Actually, there are several different versions of the rectangular rule out there. If you are interested, these are mentioned in Sections 5.1 and 5.2 of Thomas' Calculus ( $11^{\text {th }}$ edition).

### 7.3 The Trapezium Rule

Another method of calculating an integral approximately is the trapezoidal (trapezium) rule. The procedure is as follows...

Again, divide the interval $a \leq x \leq b$ into $n$ equal subintervals, i.e.

$$
a=x_{0}<x_{1}<x_{2}<\ldots<x_{n-1}<x_{n}=b
$$

each with length $h=(b-a) / n$.


Figure 7.2: The Trapezium Rule visualised. This time, the area under the curve $y=f(x)$ is approximated by a sum of $n$ trapezia, instead of rectangles.


Figure 7.3: This is the first trapezium from Figure 7.2. One way to calculate its area is to split it up into a triangle and a rectangle, calculate their areas separately, then add the two areas together.

$$
\begin{aligned}
\text { Area of first trapezium: } \begin{aligned}
\mathcal{A}_{1} & =\text { Area of rectangle }+ \text { Area of triangle } \\
& =f(a) h \quad+\frac{1}{2} h\left(f\left(x_{1}\right)-f(a)\right) \\
& =\frac{1}{2} h\left[f(a)+f\left(x_{1}\right)\right] \\
\text { Area of next trapezium: } \mathcal{A}_{2} & =\frac{1}{2} h\left[f\left(x_{1}\right)+f\left(x_{2}\right)\right]
\end{aligned}
\end{aligned}
$$

Area of penultimate trapezium: $\mathcal{A}_{n-1}=\frac{1}{2} h\left[f\left(x_{n-2}\right)+f\left(x_{n-1}\right)\right]$

$$
\text { Area of last trapezium: } \mathcal{A}_{n}=\frac{1}{2} h\left[f\left(x_{n-1}\right)+f(b)\right]
$$

Then

$$
\begin{aligned}
\mathscr{I} & =\int_{a}^{b} f(x) \mathrm{d} x \approx \text { Sum of all } n \text { trapezia } \\
& =\frac{1}{2} h\left\{f(a)+f\left(x_{1}\right)+f\left(x_{1}\right)+f\left(x_{2}\right)+f\left(x_{2}\right)+\cdots\right. \\
& \left.\cdots+f\left(x_{n-2}\right)+f\left(x_{n-2}\right)+f\left(x_{n-1}\right)+f\left(x_{n-1}\right)+f(b)\right\},
\end{aligned}
$$

i.e.

$$
\mathscr{I} \approx \frac{h}{2}\left\{f(a)+f(b)+2\left[f\left(x_{1}\right)+f\left(x_{2}\right)+\cdots+f\left(x_{n-1}\right)\right]\right\} .
$$

where

$$
h=\frac{b-a}{n} \quad x_{i}=a+i h, \quad i=1,2,3, \ldots, n-1 .
$$

Example 7.1. Estimate

$$
\mathscr{I}=\int_{1}^{2} \frac{1}{x} \mathrm{~d} x
$$

using the trapezium rule with $n=5$.
Note that we have $b=2, a=1$ and $n=5$.

$$
\therefore \quad h=\frac{b-a}{n}=\frac{2-1}{5}=\frac{1}{5}=0.2 .
$$

So

$$
a=1, x_{1}=1.2, x_{2}=1.4, x_{3}=1.6, x_{4}=1.8, b=2,
$$

and

$$
\begin{aligned}
\mathscr{I} & \approx \frac{0.2}{2}\left\{f(a)+f(b)+2\left[f\left(x_{1}\right)+f\left(x_{2}\right)+f\left(x_{3}\right)+f\left(x_{4}\right)\right]\right\} \\
& =0.1\{f(1)+f(2)+2[f(1.2)+f(1.4)+f(1.6)+f(1.8)]\} \\
& =0.1\left\{\frac{1}{1}+\frac{1}{2}+2\left[\frac{1}{1.2}+\frac{1}{1.4}+\frac{1}{1.6}+\frac{1}{1.8}\right]\right\} \\
& \approx 0.6956 . \quad \text { (4 d.p) }
\end{aligned}
$$



Figure 7.4: In the last example, we used the Trapezium Rule to estimate the area shaded in blue.

Notes:

- In the previous example, the analytical value is given by

$$
\int_{1}^{2} \frac{1}{x} \mathrm{~d} x=[\ln x]_{1}^{2}=\ln 2-\ln 1=\ln 2=0.6931 \quad \text { (4.d.p). }
$$

- If we used $n=10$, we would have

$$
\mathscr{I} \approx 0.6938
$$

which is even more accurate than using $n=5$.

## Error in using the Trapezuim Rule

Let $\hat{\mathscr{I}}$ be the trapezium approximation to $\mathscr{I}$, then we define the error $\varepsilon_{T}$ as

$$
\varepsilon_{T}=\hat{\mathscr{I}}-\mathscr{I} .
$$

Then it turns out that if

$$
\left|f^{\prime \prime}(x)\right| \leq M \quad \text { for all } x \text { with } a \leq x \leq b,
$$

then

$$
\left|\varepsilon_{T}\right| \leq M \frac{(b-a)^{3}}{12 n^{2}}
$$

Example 7.2. What is the smallest $n$ such that

$$
\mathscr{I}=\int_{0}^{2} e^{x^{2}} \mathrm{~d} x
$$

has a maximum error of 1 ?
We must choose $n$ large enough such that $\left|\varepsilon_{T}\right| \leq 1$. Note that

$$
f(x)=e^{x^{2}} \quad \Rightarrow \quad f^{\prime \prime}(x)=\left[2+4 x^{2}\right] e^{x^{2}}
$$

We are interested in $0 \leq x \leq 2$; on this interval the maximum value of $f^{\prime \prime}(x)$ occurs at $x=2$, thus $M=f^{\prime \prime}(2) \approx 983$ (rounded up). So

$$
\left|\varepsilon_{T}\right| \leq M \frac{(b-a)^{3}}{12 n^{2}} \leq 983 \frac{2^{3}}{12 n^{2}} \approx \frac{655}{n^{2}}
$$

i.e we need

$$
\frac{655}{n^{2}} \leq 1 \quad \Rightarrow \quad n^{2} \geq 655
$$

The smallest such $n$ that satisfies this is $n=26$.

### 7.4 Simpson's Rule

Simpson's Rule is yet another method of numerical integration. It is credited to Thomas Simpson (1710-1761), an English mathematician, though there is evidence that similar methods were used 100 years prior to him.

So far, we looked at two methods for numerical integration:

- Piecewise constant approximation $\Longrightarrow$ Rectangular Rule
- Piecewise linear approximation $\Longrightarrow$ Trapezium Rule
- Piecewise quadratic approximation $\Longrightarrow$ Simpson's Rule

For Simpson's rule we divide $a \leq x \leq b$ into an even number of subintervals $n$ of length $h=(b-a) / n$ with endpoints

$$
a=x_{0}<x_{1}<x_{2}<\ldots<x_{n-1}<x_{n}=b,
$$

Main idea: Suppose a typical parabola $P_{i}$ (i.e. $a x^{2}+b x+c$ ) passes through three consecutive points $\left(x_{i-1}, y_{i-1}\right),\left(x_{i}, y_{i}\right),\left(x_{i+1}, y_{i+1}\right)$.


Figure 7.5: Simpson's Rule visualised. For this method, the curve $y=f(x)$ is approximated using $n$ parabolae; then the area underneath the parabolae is taken as the approximate value of the integral.

We will not go through the derivation, but I can tell you that Simpson's formula turns out to be...

$$
\frac{h}{3}\left(\mathcal{S}_{0}+4 \mathcal{S}_{1}+2 \mathcal{S}_{2}\right)
$$

where

$$
\begin{align*}
& S_{0}=f(a)+f(b),  \tag{7.1}\\
& S_{1}=f\left(x_{1}\right)+f\left(x_{3}\right)+f\left(x_{5}\right)+\ldots+f\left(x_{n-1}\right),  \tag{7.2}\\
& S_{2}=f\left(x_{2}\right)+f\left(x_{4}\right)+f\left(x_{6}\right)+\ldots+f\left(x_{n-2}\right) . \tag{7.3}
\end{align*}
$$

Observe that for all the indices that appear in $S_{1}$, are odd, while those for $S_{2}$ are even (remember that as $n$ must be even, we have that $(n-1)$ is odd whilst $(n-2)$ is even). Meanwhile it can be shown for Simpson's rule that if

$$
\left|f^{(4)}(x)\right| \leq M \quad \text { for all } x \text { with } a \leq x \leq b,
$$

then

$$
\left|\varepsilon_{S}\right| \leq \frac{M(b-a)^{5}}{180 n^{4}}
$$

Example 7.3. Evaluate

$$
\mathscr{I}=\int_{1}^{2} \frac{1}{x} \mathrm{~d} x
$$

using Simpson's rule with $n=10, a=1, b=2$.

Note that

$$
h=\frac{2-1}{10}=\frac{1}{10}=0.1
$$

and keep track of all the values of $x_{i}$ and $f\left(x_{i}\right)$ as follows...

| $i$ | $x_{i}$ |  | $f\left(x_{i}\right)=1 / x_{i}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1.0 | 1 |  |  |
| 1 | 1.1 |  | $10 / 11$ |  |
| 2 | 1.2 |  |  | $5 / 6$ |
| 3 | 1.3 |  | $10 / 13$ |  |
| 4 | 1.4 |  |  | $5 / 7$ |
| 5 | 1.5 |  | $2 / 3$ |  |
| 6 | 1.6 |  |  | $5 / 8$ |
| 7 | 1.7 |  | $10 / 17$ |  |
| 8 | 1.8 |  |  | $5 / 9$ |
| 9 | 1.9 |  | $10 / 19$ |  |
| 10 | 2.0 | $1 / 2$ |  |  |
| Sums | 1.5000000 | 3.459539 | 2.728175 |  |

i.e.

$$
\begin{aligned}
& \mathcal{S}_{0}=1.500000 \\
& \mathcal{S}_{1}=3.459539 \\
& \mathcal{S}_{2}=2.728175
\end{aligned}
$$

and therefore

$$
\hat{\mathscr{I}}=\frac{h}{3}\left(\mathcal{S}_{0}+4 \mathcal{S}_{1}+2 \mathcal{S}_{2}\right)=0.693150
$$

Compare with the exact value

$$
\mathscr{I}=\int_{1}^{2} \frac{\mathrm{~d} x}{x}=\ln 2=0.69314718
$$

hence this is correct to FIVE d.p. (Trapezium Rule was correct to 1 d.p.)

### 7.5 Newton's Method for Root-Finding

In engineering, it is often required to find $x$ such that

$$
\begin{equation*}
f(x)=0 \tag{7.5}
\end{equation*}
$$

These values of $x$ are known as roots of $f(x)$.
Examples:

1) $x^{2}-3 x+2=0$
2) $\sin x=\frac{1}{2} x$
3) $\cosh x \cos x=-1$

All of these can be written in the form 7.5 .
In this course, I will introduce you to one of the fastest methods for finding roots of $f(x) \ldots$ Newton's Method (a.k.a. Newton-Raphson Method).

## How the method works:

Let our first (initial) guess to the root be $x_{0}$. Then $x_{1}$ is the point where the tangent to the curve $f$ at $x_{0}$ intersects the $x$-axis.

$$
\tan \beta=f^{\prime}\left(x_{0}\right)=\frac{f\left(x_{0}\right)}{x_{0}-x_{1}},
$$

i.e.

$$
x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}
$$

Now $x_{1}$ is our new guess for the root of $f(x)$. But we might want a better guess; call this $x_{2}$. It turns out the next iteration is

$$
x_{2}=x_{1}-\frac{f\left(x_{1}\right)}{f^{\prime}\left(x_{1}\right)}
$$

and we can repeat the procedure yet again:

$$
x_{3}=x_{2}-\frac{f\left(x_{2}\right)}{f^{\prime}\left(x_{2}\right)}
$$

and so on. We can keep iterating until we get the desired accuracy, using the formula:

$$
x_{n+1}=x_{n}-\frac{f\left(x_{n}\right)}{f^{\prime}\left(x_{n}\right)}
$$

Example 7.4. Find the positive solution of

$$
2 \sin x=x
$$

First, get the original equation into the form $f(x)=0$ :

$$
\begin{aligned}
f(x) & =x-2 \sin x \\
\Rightarrow f^{\prime}(x) & =1-2 \cos x
\end{aligned}
$$



Figure 7.6: In the last example, we used the Trapezium Rule to estimate the area shaded in blue.


Figure 7.7: A plot of $y=2 \sin x$ and $y=x$. The root we are after is the positive $x$-value at the point where the two functions intersect.

Then, by Newton's Method,

$$
\begin{aligned}
x_{n+1} & =x_{n}-\frac{x_{n}-2 \sin x_{n}}{1-2 \cos x_{n}} \\
& =\frac{2\left(\sin x_{n}-x_{n} \cos x_{n}\right)}{1-2 \cos x_{n}}=\frac{N_{n}}{D_{n}}
\end{aligned}
$$

We need an initial guess, e.g. $x_{0}=2$.

| $n$ | $x_{n}$ | $N_{n}$ | $D_{n}$ | $x_{n+1}=N_{n} / D_{n}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 2.00 | 3.483 | 1.832 | 1.901 |
| 1 | 1.901 | 3.125 | 1.648 | 1.896 |
| 2 | 1.896 | 3.107 | 1.639 | 1.896 |

The actual solution to 4 d.p is 1.8955 .

Advantages of Newton's method:

- Converges very fast!
- You only need to give one initial guess (some methods require TWO).

Disadvantages:

- You need to calculate the derivative of $f(x)$.
- Sometimes the method doesn't converge to a root at all!
- The method is useless if your first guess is a stationary point of $f(x)$ (because you get a division by zero).


## Chapter 8

## Probability and Statistics

### 8.1 Basic Probability

For an event $E$, the probability of the $E$ occurring, denoted $\mathrm{P}(E)$, is a number such that

$$
0 \leq \mathrm{P}(E) \leq 1
$$

where

$$
\begin{aligned}
& \mathrm{P}(E)=0 \quad \Longrightarrow \quad E \quad \text { is impossible, } \\
& \mathrm{P}(E)=1 \Longrightarrow E \quad \text { is certain. }
\end{aligned}
$$

Example 8.1 (Rolling a die). The set of all possible outcomes is the sample space, denoted $S$, i.e.

$$
S=\{1,2,3,4,5,6\}
$$

Let $A$ be the event of getting an even number in one roll. Then we have

$$
A=\{2,4,6\}
$$

and therefore

$$
\mathrm{P}(A)=\frac{3}{6}=\frac{1}{2}
$$

Example 8.2. We randomly select 2 lightbulbs from a set of 5 bulbs (numbered 1 to 5 ). The sample space consists of 10 possible outcomes:

$$
\begin{aligned}
S= & \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\} \\
& \{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}
\end{aligned}
$$

Note that $|S|=10$ is the number of elements in $S$, also known as the cardinality of the set $S$. We may be interested in the following events:

A: No faulty bulbs
B: One faulty bulb
C: Two faulty bulbs

Now assume that bulbs 1,2 and 3 are all faulty. We see that event $A$ occurs only if we draw bulbs 4 and 5 (i.e. outcome $\{4,5\}$ ).

$$
\therefore \quad \mathrm{P}(A)=\frac{1}{10} .
$$

Event B occurs if we draw $\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\}$ or $\{3,5\}$. Hence

$$
\mathrm{P}(B)=\frac{6}{10} .
$$

Meanwhile, Event C occurs if we draw $\{1,2\},\{1,3\},\{2,3\}$, and therefore

$$
\mathrm{P}(C)=\frac{3}{10} .
$$

Definition 8.1. The set of all elements (outcomes) not in $E$ in the sample space $S$ is called the compliment of $E$, usually denoted $E^{c}$ or $\bar{E}$.

Example 8.3. $E=$ randomly rolled die gives an even number, i.e.

$$
E=\{2,4,6\}
$$

then $E^{c}=$ randomly rolled die gives an odd number, i.e.

$$
E^{c}=\{1,3,5\} .
$$

Let $A$ and $B$ be two events in an experiment.
Definition 8.2. The event consisting of all the elements of the sample space that belong to either $A$ or $B$ is called the union of $A$ and $B$ and is denoted as $A \cup B$.


Figure 8.1: $A$ Venn diagram. The union $A \cap B$ is shaded in green.
Definition 8.3. The event consisting of all the elements of the sample space that belong to both $A$ and $B$ is called the intersection of $A$ and $B$ and is denoted as $A \cap B$.

Example 8.4. Suppose that we are rolling a die, then consider the following events:

A: The die gives a number not smaller than 4 .
B: The die gives a number that is a multiple of 3 .


Figure 8.2: $A$ Venn diagram. The intersection $A \cup B$ is shaded in green.

$$
A=\{4,5,6\}, \quad B=\{3,6\},
$$

then

$$
A \cup B=\{3,4,5,6\}, \quad A \cap B=\{6\}
$$

Definition 8.4. Events $A$ and $B$ are said to be mutually exclusive events if they have no element in common, i.e. if

$$
A \cap B=\{ \}=\emptyset
$$

where the symbol $\emptyset$ denotes the empty set. It has no elements, so the cardinality of the empty set is zero.

## The Axioms of Probability

1. For any event $E$ in a sample space $S$,

$$
0 \leq \mathrm{P}(E) \leq 1
$$

2. For the entire sample space $S$, we have $\mathrm{P}(S)=1$.
3. If $A$ and $B$ are mutually exclusive events, then

$$
\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)
$$

Fact: If $A$ and $B$ are any events, then

$$
\mathrm{P}(A \cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)
$$

## Fact:

$$
\mathrm{P}(E)=1-\mathrm{P}\left(E^{c}\right)
$$

i.e. the probability of $E$ occurring is 1 - (the probability of $E$ not occurring).

Example 8.5 (Rolling a die again!). The event space is

$$
S=\{1,2,3,4,5,6\}
$$

with $\mathrm{P}(1)=1 / 6, \mathrm{P}(2)=1 / 6$, etc.

A: The event that an even number is given.

$$
\mathrm{P}(A)=\mathrm{P}(2)+\mathrm{P}(4)+\mathrm{P}(6)=\frac{1}{2}
$$

B: The event that a number greater than 4 turns up.

$$
\mathrm{P}(B)=\mathrm{P}(5)+\mathrm{P}(6)=\frac{1}{3}
$$

Example 8.6. Five coins are tossed simultaneously. What is the probability of obtaining at least one head?

Note: There are in total $2^{5}=32$ possible outcomes, only one of which has no heads. Therefore

$$
\begin{aligned}
\mathrm{P}(\text { At least one head }) & =1-\mathrm{P}(\text { No heads }) \\
& =1-\frac{1}{32}=\frac{31}{32}
\end{aligned}
$$

Example 8.7. The probability that a person watches TV is $\mathrm{P}(T)=0.6$; the probability that the same person listens to the radio $\mathrm{P}(R)=0.3$. The probability that they do both is 0.15. What is the probability that they do neither?

Using the addition law,

$$
\begin{gathered}
\mathrm{P}(T \cup R)=\mathrm{P}(T)+\mathrm{P}(R)-\mathrm{P}(T \cap R) \\
=0.6+0.3-0.15=0.75 \\
\therefore \quad \mathrm{P}(\text { They do neither })=1-\mathrm{P}(T \cup R)=0.25 .
\end{gathered}
$$

## Conditional probability

Often it is required to find the probability of an event $B$ given that an event $A$ has already occurred. This is known as the conditional probability of $B$ given $A$, and is denoted $\mathrm{P}(B \mid A)$. The intuition behind this is that $A$ gives a "reduced sample space", and therefore

$$
\mathrm{P}(B \mid A)=\frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}
$$

Example 8.8 (Conditional Probability). The probability $\mathrm{P}(A)$ that it rains in Manchester on July $15^{\text {th }}$ is 0.6 , while the probability $\mathrm{P}(A \cap B)$ that it rains there on both the $15^{\text {th }}$ and $16^{\text {th }}$ is 0.35 . Given that it rains on the $15^{\text {th }}$, what is the probability that it rains on the next day?

Note: $B$ is the event that it rains in Manchester on July $16^{\text {th }}$. We need to find $\mathrm{P}(B \mid A)$, and using the formula for conditional probability :

$$
\mathrm{P}(B \mid A)=\frac{\mathrm{P} A \cap B}{\mathrm{P}(A)}=\frac{0.35}{0.6}=\frac{7}{12}=0.583 . \quad(3 \mathrm{~d} . \mathrm{p})
$$

Example 8.9. A fridge contains 10 cans of lager, three of which are " 4 X " (to be avoided). Robb selects 2 cans at random. Find the probability that none of the selected cans are " 4 X ".

$$
\text { Let } \begin{aligned}
A & =\text { First can selected is not } 4 \mathrm{X}, \\
& B=\text { Second can selected is not } 4 \mathrm{X} .
\end{aligned}
$$

We will look at two different cases. .

1 The case with replacement, i.e. Robb puts the first can back in the fridge before choosing the second. Then

$$
\mathrm{P}(A)=\mathrm{P}(B)=\frac{7}{10}
$$

and

$$
\mathrm{P}(A \cap B)=\frac{7}{10} \times \frac{7}{10}=0.49
$$

2 Sampling without replacement, i.e. the first can is NOT put back in the fridge. Then. . .

$$
\begin{aligned}
& \mathrm{P}(A)=\frac{7}{10}, \quad \text { and } \quad \mathrm{P}(B \mid A)=\frac{7 \times 1}{10 \times 1}=\frac{6}{9}=\frac{2}{3} \\
\therefore \quad & \mathrm{P}(A \cap B)=\mathrm{P}(A) \mathrm{P}(B \mid A)=\frac{7}{10} \times \frac{2}{3}=\frac{14}{30} \approx 0.47
\end{aligned}
$$

### 8.2 Random Variables

Sometimes engineers must work with a variable $X$ whose (real) value is subject to variations due to chance (randomness). We call $X$ a random variable.

So $X$ can take on a set of possible different values, each with a corresponding probability. We can say that for each possible value $a$, for

$$
X=a \quad \text { the probability of this value is } \quad \mathrm{P}(X=a)
$$

We can then say that the probability that $X$ assumes any value within the range:

1. $b<X<c$ is $\mathrm{P}(b<X<c)$
2. $X \leq c$ is $\mathrm{P}(X \leq c)$
3. $X>c$ is $\mathrm{P}(X>c)$.

Actually,

$$
\mathrm{P}(X \leq c)+\mathrm{P}(X>c)=\mathrm{P}(\text { All possible values of } X)=1
$$

or equivalently,

$$
\mathrm{P}(X>c)=1-\mathrm{P}(X \leq c)
$$

Example 8.10. Let

$$
X=\text { Score obtained when I roll a fair die.. }
$$

Then...

$$
\begin{array}{lr}
\mathrm{P}(X=1)=\frac{1}{6}, & \mathrm{P}(1 \leq X \leq 2)=\frac{1}{3}, \\
\mathrm{P}(1<X<2)=0, & \mathrm{P}(X<0.5)=0 .
\end{array}
$$

In this example, our random variable is discrete. Random variables can also be continuous, but we will only discuss discrete ones in this course.

Let $x_{1}, x_{2}, \ldots$ be the possible values of $X$, each with probabilities $\mathrm{P}_{1}, \mathrm{P}_{2}, \ldots$

Then we can consider a probability distribution function (p.d.f) for $f(x)$.
Note that the condition $\sum_{j} f\left(x_{j}\right)=\sum_{j} \mathrm{P}_{j}=1$ is necessary.
Example 8.11 (Rolling one die). By sketching the p.d.f, we can visualise the distribution of the random variable $X \ldots$


Figure 8.3: The p.d.f. for rolling one die. Observe that the probabilities for the scores 1 to 6 are all the same (and add up to one). Moreover, the p.d.f. shows that there is no chance of scoring 7, 8, 9, ...

This particular example is a uniformly distributed random variable.
Example 8.12 (Rolling two dice). There are 36 possible outcomes, all with a probability of $\frac{1}{36}$. Let's define the random variable $X$ as:

$$
X=\text { Sum of the numbers obtained by rolling two dice. }
$$

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(x)$ | $\frac{1}{36}$ | $\frac{2}{36}$ | $\frac{3}{36}$ | $\frac{4}{36}$ | $\frac{5}{36}$ | $\frac{6}{36}$ | $\frac{5}{36}$ | $\frac{4}{36}$ | $\frac{3}{36}$ | $\frac{2}{36}$ | $\frac{1}{36}$ |

Example 8.13. Suppose $X=\{0,1,2,3\}$, and the following two distributions are:


Figure 8.4: The p.d.f. for rolling two dice. Unsurprisingly, there is zero chance of gaining a sum of thirteen!
i $f(x)=\frac{1}{8}(1+x)$
ii $f(x)=\frac{1}{10}(1+x)$.

Only one of these is a valid p.d.f. Which one, and why?
Answer: (ii) is valid, but (i) is not.
Why: Need $\sum_{j} f\left(x_{j}\right)=1$. Only (ii) satisfies it.

Definition The mean, expectation or expected value $\mu$ of a discrete p.d.f:

$$
\left[(\mathrm{E}(X)=] \quad \mu=\sum_{j} x_{j} f\left(x_{j}\right)=x_{1} f\left(x_{1}\right)+x_{2} f\left(x_{2}\right)+\cdots\right.
$$

Example 8.14 (Expected value for rolling a fair die). Recall that

$$
\begin{aligned}
& f\left(x_{j}\right)=\frac{1}{6} \quad \text { when } \quad j=1,2, \ldots, 6 \\
\Rightarrow \quad & \mu=1 \times \frac{1}{6}+2 \times \frac{2}{6}+\ldots+6 \times \frac{6}{6}=3.5
\end{aligned}
$$

Granted, we can't gain a score of 3.5 if we roll the die only once. But that is not what $\mu$ means. Actually, $\mu$ represents the average "score" you would get if you rolled the die many times.

Example 8.15. A stranger shows you a game where you draw a ball out of a bag. There are 6 white balls and 4 blue balls in the bag.

- If the ball is white, you win 40 p.
- If the ball is blue, you lose 80 p.

Afterwards, the ball is replaced. What are your expected winnings? And is it worth playing that game?

Let $X=$ winnings obtained after drawing the ball out.

$$
\begin{aligned}
& \text { When } X=40 \\
& X\left(x_{1}\right) \\
& \mathrm{P}\left(x_{1}\right)=\frac{6}{10} \\
& X\left(x_{2}\right) \\
& \mathrm{P}\left(x_{2}\right)=\frac{4}{10}
\end{aligned}
$$

Therefore for the expected value

$$
\Rightarrow \quad \mu=x_{1} \mathrm{P}\left(x_{1}\right)+x_{2} \mathrm{P}\left(x_{2}\right)=40 \times \frac{6}{10}+(-80) \times \frac{4}{10}=-8 \mathrm{p}
$$

$\therefore$ After playing $n$ games you can expect to lose $8 n$ pence!
Better off to NOT play this game.

Definition The variance of a distribution, denoted $\sigma^{2}($ or $\operatorname{Var}(X))$ is defined by

$$
\begin{aligned}
\sigma^{2}=\operatorname{Var}(X) & =\sum_{j}\left(x_{j}-\mu\right)^{2} f\left(x_{j}\right) \\
& =\left(x_{1}-\mu\right)^{2} f\left(x_{1}\right)+\left(x_{2}-\mu\right)^{2} f\left(x_{2}\right)+\cdots
\end{aligned}
$$

Shortcut: $\sigma^{2}=\mathrm{E}\left(X^{2}\right)-\mu^{2}$, where $\mathrm{E}\left(X^{2}\right)$ is the mean for $X^{2}$.

$$
\left[\mathrm{E}\left(X^{2}\right)=\sum_{j} f\left(x_{j}\right) x_{j}^{2}\right]
$$

Can interpret $\sigma^{2}$ as a measure of the spread of the data. Specifically, it is the expected square deviation of $X$ from the mean $\mu$.

Example 8.16 (Coin toss). Let 1 and 0 denote heads and tails respectively. It is easy to show that

$$
\mu=0 \times \frac{1}{2}+1 \times \frac{1}{2}=\frac{1}{2}
$$

but what is the variance?
Take the shortcut...

$$
\sigma^{2}=\left(0^{2} \times \frac{1}{2}+1^{2} \times \frac{1}{2}\right)-\left(\frac{1}{2}\right)^{2}=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}
$$

### 8.3 The Binomial Distribution

Start by conducting an experiment (trial) with only two outcomes. They can be labelled "success" or "failure", and their repective probabilities are $p$ and $q=1-p$.
E.g. Scoring a 6 from a die roll: $p=\frac{1}{6}, q=\frac{5}{6}$.

Then if the trial is repeated a fixed number of times $(n)$, define a new discrete random variable:

$$
X=\text { Number of successes in } n \text { trials. }
$$

We assume four conditions:

1. The trial must only have two outcomes
2. Fixed number of trials
3. The probability of success must be the same for all trials
4. The trials are independent.

Example 8.17. Find the probability of $0,1,2,3,4$ successes in an experiment consisting of up to 4 repeated trials with probability of success $p(\therefore q=1-p)$.

| Number of Trials <br> Number of Successes | 1 | 2 | 3 | 4 |
| :---: | :--- | :--- | :--- | :--- |
| 0 | $q$ | $q^{2}$ | $q^{3}$ | $q^{4}$ |
| 1 | $p$ | $2 p q$ | $3 p q^{2}$ | $4 p q^{3}$ |
| 2 | 0 | $p^{2}$ | $3 p^{2} q$ | $6 p^{2} q^{2}$ |
| 3 | 0 | 0 | $p^{3}$ | $4 p^{3} q$ |
| 4 | 0 | 0 | 0 | $p^{4}$ |

Generally, we can consider the p.d.f. $f(x)=\mathrm{P}(X=x)$. Then the probability of $x$ successes in $n$ trials is

$$
\mathrm{P}(X=x)=\binom{n}{x} p^{x} q^{n-x},
$$

where $\binom{n}{x}$ is the binomial coefficient, and the p.d.f. corresponds to the Binomial Distribution.

Recall that

$$
\binom{n}{x}=\frac{n!}{x!(n-x)!}
$$

These binomial coefficients represent the number of ways of choosing $x$ objects from a set of $n$ objects.

Example 8.18. We roll a die 56 times. What is the probability of getting at least three sixes?

Define a random variable $X$ as

$$
X=\text { Number of sixes thrown in } 56 \text { trials. }
$$

Then we say that

$$
X \sim \mathrm{~B}\left(n=56, p=\frac{1}{6}\right)
$$

Then we want

$$
\begin{aligned}
& \mathrm{P}(\geq 3)=1-\mathrm{P}(X=0,1 \text { or } 2) \\
& 1-\left[\left(\frac{5}{6}\right)^{56}+\binom{56}{1}\left(\frac{5}{6}\right)^{55}\left(\frac{1}{6}\right)+\binom{56}{2}\left(\frac{5}{6}\right)^{54}\left(\frac{1}{6}\right)^{2}\right]
\end{aligned}
$$

Note: It is perfectly fine to leave your answer in this form!

Example 8.19. A factory produces plenty of board pens. However, $10 \%$ of the pens are defective. If I open a random box containing twenty board pens, what is the probability that:
i Exactly 3 pens are defective?
ii More than 3 pens are defective?
(Answer to 3 decimal places)
First, if $X=$ number of faulty pens in a box of 20,

$$
X \sim \mathrm{~B}(20,0.1)
$$

i We want

$$
\mathrm{P}(X=3)=\binom{20}{3}(0.1)^{3}(0.9)^{17} \approx 0.190
$$

ii This is $\mathrm{P}(X \geq 3)$, i.e.

$$
\begin{aligned}
\mathrm{P}(X \geq 3) & =1-\mathrm{P}(X \leq 2) \\
& =1-\left[0.9^{20}+\binom{20}{1}(0.1)(0.9)^{19}+\binom{20}{2}(0.1)^{2}(0.9)^{18}\right] \\
& \approx 0.323
\end{aligned}
$$

## Mean and variance of $\mathrm{B}(n, p)$

Since

$$
f(x)=\binom{n}{x} p^{x} q^{1-x}
$$

it turns out that

$$
\begin{aligned}
\quad \text { Mean: } & \mu=\sum_{x=0}^{n} x f(x)=\sum_{x=0}^{n}\binom{n}{x} p^{x} q^{n-x} x=n p \\
\text { Variance: } & \sigma^{2}=n p q=n p(1-p) .
\end{aligned}
$$

So for the board pen example, $\mu=2, \sigma^{2}=1.8$.

### 8.4 The Poisson Distribution

Consider the following scenarios:
i Number of phone calls arriving at a call centre per hour.
ii Number of cars crossing a bridge per hour.
iii Number of faults in a length of cable.

These problems require a distribution that involves an average rate $\mu$. Actually, there is one - it is the Poisson distribution, and its p.d.f. is:

$$
\mathrm{P}(X=x)=\frac{e^{-\mu} \mu^{x}}{x!}
$$

where $X=0,1,2, \ldots$, to $\infty$.
Example 8.20. On average, 240 cars per hour pass a check point, and a queue forms if more than three cars pass through in a given minute.

What is the probability of a queue forming in a randomly selected minute?

$$
\text { Average number of cars per minute }=\frac{240}{60}=4=\mu .
$$

Let

$$
X=\text { Number of cars passing at a randomly selected minute. }
$$

Then $X \sim \operatorname{Po}(4)$, and we require

$$
\begin{aligned}
& \mathrm{P}(X \geq 3)=1-\mathrm{P}(0 \leq X \leq 3) \\
& \quad 1-[\mathrm{P}(X=0)+\mathrm{P}(X=1)+\mathrm{P}(X=2)+\mathrm{P}(X=3)] \\
& \quad=1-0.4331=0.5669 .
\end{aligned}
$$

One important use of the Poisson distribution is to APPROXIMATE the Binomial distribution, because Poisson is easier to compute.

Recall that for binomial,

$$
f(x)=\binom{n}{x} p^{x} q^{n-x}
$$

Then if you let $p \longrightarrow 0$ and $n \longrightarrow \infty$ with $\mu=n p$ fixed and finite,

$$
f(x) \longrightarrow \operatorname{Po}(\mu) .
$$

Moreover, the Poisson distribution has mean $\mu$ and variance $\mu$.
Example 8.21. A factory produces screws. The probability that a randomly selected screw is defective is given by $p=0.01$.

In a random sample of 100 screws, what is the probability that there will be more than two defective screws?

$$
\begin{aligned}
& \text { Let } A=\text { More than two defective screws } \\
& \Rightarrow \quad A^{C}=\text { At most } 1 \text { defective. } \\
& \begin{aligned}
\mathrm{P}\left(A^{C}\right) & =\binom{100}{0}(0.01)^{0}(0.99)^{100}+\binom{100}{1}(0.01)^{1}(0.99)^{99} \\
& +\binom{100}{2}(0.01)^{2}(0.99)^{98} .
\end{aligned}
\end{aligned}
$$

After spending ages on your calculator, you finally get

$$
\begin{equation*}
\Rightarrow \mathrm{P}(A)=1-\mathrm{P}\left(A^{C}\right) \approx 0.0794 \tag{3s.f.}
\end{equation*}
$$

Alternative: Poisson approximation. As $n$ is large and $p$ small, we have

$$
\begin{gathered}
\mu=n p=1, \quad \therefore 1 \text { out of } 100 \text { defective on average. } \\
\Rightarrow \quad \mathrm{P}\left(A^{C}\right) \approx e^{-1}\left(\frac{1^{0}}{0!}+\frac{1^{1}}{1!}+\frac{1^{2}}{2!}\right)=\times \frac{5}{2} e^{-1} \approx 0.9197
\end{gathered}
$$

and

$$
\mathrm{P}(A)=1-\mathrm{P}\left(A^{C}\right) \approx 0.0803 . \quad \text { Close to the binomial result! }
$$

