EC3014: Game Theory, Lecture 2

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Mixed Strategy Nash Equilibria

Goalie

Kicker		L	R
	L	$2, 8$	$9, 1$
	R	$8, 2$	$3, 7$

Penalty Kick Game

this is an example of a constant sum game (or zero-sum game)

why: if we subtract a fixed number from every payoff for a player, this does not change the strategic situation

So we can subtract 5 from each player's payoff to make it zero sum

No Nash equilibrium in pure strategies.

Mixed extension of the game: allow players to choose probabilities for each pure strategies, such that

a) each probability is positive

b) sum of probabalities is one

Let p be the probability that kicker plays L.

Let q be prob that goalie plays L.

The expected utility of player 1 from L given q is

$$u_1(L, q) = 2q + 9(1 - q)$$

$$u_1(R, q) = 8q + 3(1 - q)$$

$$u_1(p, q) = pu_1(L, q) + (1 - p)u_1(R, q)$$

Expression is linear in p

If $u_1(L, q) > u_1(R, q)$, then above is strictly increasing in p, and only $p = 1$ can be optimal.

If $u_1(L, q) < u_1(R, q)$, then above is strictly dereasing in p, and only $p = 0$ can be optimal.

Interior value of p can only be optimal if $u_1(L, q) = u_1(R, q)$, in this case, *any* value of p is optimal.

If $q = \frac{1}{2}$, then $u_1(L, q) = u_1(R, q)$.

$$u_2(L, p) = 8p + 2(1 - p)$$

$$u_2(R, q) = p + 7(1 - p)$$

If $p = \frac{5}{12}$, then $u_2(L, p) = u_2(R, p)$.

Game has Nash equilibrium in mixed strategies, $(p = \frac{5}{12}, q = \frac{1}{2})$.

Key points:

In a mixed strategy equilibrium, a player who randomizes between two strategies is indifferent between all the strategies that she asssigns positive probability to.

Player 1's mixed strategy is chosen in order to make player 2 indifferent, and vice versa.

We allow players to play any mixed strategy, i.e. assign probabilities to each pure strategy

If player i has k pure strategies, then a mixed strategy is a vector $\mathbf{p}$ =$(p_i^1, p_i^2, ..., p_i^k)$

$p_i^j \geq 0, \sum_{j=1}^{k_i} p_i^j = 1$.

Theorem (Nash): Let G be a game with finitely many players, where each player's strategy set is finite. G has a Nash equilibrium, possibly in mixed strategies.

Let $m = (m_1, m_2, ..., m_n)$ be a Nash equilibrium.

The *support* of m_i is the set of pure strategies j : $p_i^j > 0$.

Then for every player i :

1. all strategies in the support of m_i have the same payoff against m_{-i}.

2. any strategy that is not in the support of m_i has a (weakly) lower payoff.

Empirical Evidence

Walker and Wooder's, Minmax play at Wimbledon, AER 2001

Server seeks to maximize probability of winning this point.

Server can serve L or R

Receiver can anticipate L or R

Each strategy combination $(L, L), (L, R), (R, L), (R, R)$ determines a probability of winning the point.

2×2 game with unique Nash equilibrium in mixed strategies

This is a zero sum game, and Nash equilibrium strategies are also know as *minmax* strategies

Point game (payoff matrix) varies according who is serving, and also between deuce court and ad court.

Assume that this payoff matrix does not change otherwise

i.e. every game where Federer is serving to Murray on ad court is the same over the entire match.

We therefore observe repeated plays of the this same game over the match.

Observe direction of serve (L or R)

Do not observe direction of anticipation by receiver.

Testable Prediction:

Fix one of the 4 point games in the match.

$$\text{Pr(win| serves } L) = \text{ Pr(win| serves } R) = p$$

Outcomes when serve L are independent draws from a Bernoulli trial with success probability $p(W|L)$

Outcomes when serve R are independent draws from a Bernoulli trial with success probability $p(W|R = p(W|L)$

Statistical test whether actual winning frequencies are significantly different from each other.

In any match, 4 different point games

10 grand slam matches – 40 different point games.

Since p is not known the Pearson test statistic is distributed as χ^2 with one degree of freedom.

If we test in 40 games and reject at 5% level, we should expect 2 rejections given that the null (theory) is true.

one rejection (Sampras v Agassi, 1995)

the p values in each of the 40 tests should be 40 draws from uniform distribution on $[0, 1]$

Kolmogorov Smirnov test of this – not rejected.

2nd Testable Prediction:

Server's choices must be serially independent in each point game.

Look at the number of runs

e.g. in 6 element sequence $L, R, R, R, L, L,$ there are 3 runs.

If there are n serves, look at the number of runs k

If k is too large – too many changes of direction in serve – negative correlation.

If k is too small – runs are very long – too few changes of direction, positive correlation.

Rejection of serial independence in 5 out of 40 point games.

Overall, too many changes of direction relative to independence.

Psychological literature – subjects have difficulty generating random sequences.

Belief in Law of small numbers

Reasonably good support for predictions of mixed equilibrium in tennis serve.

Experimental work – O'Neill (1987)

Subjects played zero sum game with unique mixed equilibrium, paid according to outcomes.

Behavior quite far from predictions of equilibrium.

Inexperience. Lack of adequate incentives.

Penalty kicks in soccer

Chiappori, Levitt and Greseclose (2003, AER) and Palacios Huertas (Rev Econ Studies 2003)

Can observe direction of kick and also direction of movement of goalkeeper.

Allow for 3 different possibilities, L, R and also C, for each player.

But only few observations any single pair, unlike tennis.

Not possible to aggregate across pairs.

Tests have low power.

Aggregation bias possible.