C31: Game Theory. Affiliate Exam, December 2005: Answers

1. a) Consider a game in strategic form, G. Define a strictly dominated strategy. Show that if player i has a strictly dominated strategy s_i, then s_i cannot be played in any Nash equilibrium of the game G.

b) Consider a Cournot oligopoly with n firms, producing a homogeneous product. The market price $P = 100 - Q$, where Q is total quantity. Firms are symmetric and have zero costs. Consider the game where firms simultaneously choose quantities, where a quantity is any non-negative number, and seek to maximize expected profits.

i) what are the strictly dominated strategies for a firm?

ii) Solve for a symmetric Nash equilibrium. How does the market price in this equilibrium vary with n, the number of firms?

iii) Show or explain why there cannot be any asymmetric Nash equilibrium in this game.

a) A strategy s_i is strictly dominated if there exists some other (mixed) strategy for player i, s_i' such that:

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i}), \text{ for all } s_{-i} \in S_{-i}. \tag{1}$$

Let (s_i^*, s_{-i}^*) be a Nash equilibrium. This implies

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*), \text{ for any } s_i^{'} \in S_i. \tag{2}$$

But if s_i^* is strictly dominated, then from (1), it follows that $u_i(s_i^*, s_{-i}^*) < u_i(s_i', s_{-i}^*)$, which contradicts (2).

b) Let Q_{-i} be the total quantity produced by all firms excluding firm i. Firm i's profits are given by

$$\pi_i(q_i, Q_{-i}) = q_i(100 - Q_{-i} - q_i). \tag{3}$$

$$\frac{\partial \pi_i}{\partial q_i} = 100 - Q_{-i} - 2q_i. \tag{4}$$

Equation (4) tells us that $\frac{\partial \pi_i}{\partial q_i} < 0$ if $q_i > 50$, irrespective of the value of Q_{-i}, as long as its non-negative. Therefore any output level $q_i > 50$ is dominated by $q_i = 50$, since the latter gives a strictly higher payoff no matter what the value of Q_{-i}. So the strictly dominated strategies are $q_{i:}q_i > 50$.

ii) The first order condition yields

$$\hat{q}_i = \frac{100 - Q_{-i}}{2}. \tag{5}$$

In a symmetric equilibrium, $Q^*_{-i} = (n-1)q^*$, which yields $q^* = \frac{100}{n+1}$.

iii) Suppose that firms 1 and 2 produce different outputs in equilibrium, q_1 and q_2. From the first order condition for firm 1 we have

$$2q_1 + q_2 + \sum_{j=3}^{n} q_j = 100. \tag{6}$$

From the first order condition for firm 2 we have

$$2q_2 + q_1 + \sum_{j=3}^{n} q_j = 100. \tag{7}$$

Subtracting these two equations from each other, we get:

$$q_{1-}q_2 = 0. \tag{8}$$

This proves that there cannot be an asymmetric equilibrium.

Q2: Answers will be provided separately.

3. Consider the following Bayesian game. Nature chooses between states ω and ω', where ω is chosen with probability π. Player 1 observes the state, while player 2 has no information regarding nature's choice. The two players then play a simultaneous move game with payoffs as given below, where player 1 chooses between T and B, and 2 chooses between L and R.

	L	R
T	2,2	1,3
B	3,1	0,0

payoffs at ω

	L	R
T	5,5	0,0
B	0,0	2,2

payoffs at ω'

a) Solve for the pure strategy Bayesian Nash equilibrium of this game when $\pi = 0.8$ and $\pi = 0.2$.(There may or may not be more than one equilibrium.)

Let a be the action played by player 2 at his single information set. At state ω', the game played is a coordination game, and so player 1 would like to "match" a, i.e. she would like to play T if $a = L$, and B if $a = R$.

Suppose that $a = L$. In this case, player 1 will play B at ω (since $3 > 2$). So the candidate equilibrium is $(B, T; L)$ (i.e. 1 plays B at ω, T at ω' and 2 plays L). By construction, 1 is playing optimally given 2's strategy. 2's payoff from L at this strategy profile is given by

$$\pi.1 + (1-\pi)5 = 5 - 4\pi. \tag{9}$$

While 2's payoff from R is given by

$$\pi.0 + (1-\pi)0 = 0. \tag{10}$$

So this is an equilibrium for any value of π.

Suppose now that $a = R$. In this case, player 1 will play T at ω (since $1 > 0$). So the candidate equilibrium is $(T, B; R)$. By construction, 1 is playing optimally given 2's strategy. 2's payoff from R at this strategy profile is given by

$$\pi.3 + (1 - \pi)2 = 2 + \pi. \tag{11}$$

While 2's payoff from L is given by

$$\pi.2 + (1 - \pi)0 = 2\pi. \tag{12}$$

Since $2\pi \leq 2 + \pi$ for any π, this is an equilibrium for any value of π.

So for both values of π, we have two pure strategy equilibria.

b) Consider a different game where neither player observes the realized state. Solve for the pure strategy Nash equilibria of this game.

The game is now symmetric and the expected payoffs of the players are now given by

	L	R
T	5-3π,5-3π	$\pi, 3\pi$
B	$3\pi, \pi$	2-2π,2-2π

(T, L) is a Nash equilibrium provided that

$$5 - 3\pi \geq 3\pi \ , \tag{13}$$

i.e. as long as $\pi \leq \frac{5}{6}$. So this is an equilibrium for both values of π in the question. This also implies (B, L) or (T, R) cannot be an equilibrium for these values of π, since in the first case, 1 wants to deviate to T, and in the second case, 2 wants to deviate to L.

(B, R) is a Nash equilibrium provided that

$$2 - 2\pi \geq \pi, \tag{14}$$

i..e as long as $\pi \leq \frac{2}{3}$. So this is an equilibrium if $\pi = 0.2$, but not if $\pi = 0.8$.

To summarize: when $\pi = 0.8$, there is a unique pure strategy equilibrium, (T, L). When $\pi = 0.2$, both (T, L) and (B, R) are Nash equilibria.

4. Two players, A and B, have to decide how to divide two cakes, X and Y. Each case is of size one, and the A's payoff from a share (x, y) (of x amount of X and y amount of Y) is given by the utility function

$$\mathbf{U(x, y) = x + \boldsymbol{\lambda} y}.$$

B's payoff from a share (x, y) is given by the utility function

$$\mathbf{V(x,y) = x + \boldsymbol{\delta} y}.$$

Assume that $\lambda > 0$, **and** $\delta > \lambda$.

The mechanism for dividing the cake is as follows. A divides each of the cakes, and puts them in two bundles, so that one bundle is (x, y) **and the second bundle is** $(1-x, 1-y)$, **where** $0 \leq x \leq 1$ **and** $0 \leq y \leq 1$. **B chooses one of the two bundles, and the remaining bundle is consumed by** A.

a) Solve for a backwards induction equilibrium. You may assume that if B is indifferent between two bundles, he chooses the bundle that A would like him to choose.

Let bundle 1 be (x, y), so that bundle 2 is $(1-x, 1-y)$. Let A choose the bundles so that

i) B always finds it optimal to choose bundle 1, and

ii) Bundle 2 maximizes A's payoff subject to this constraint.

Thus the maximization problem for A can be written as:

$$\max_{x,y} \{(1-x) + \lambda(1-y)\} \tag{15}$$

subject to the constraints:

$$x + \delta y \geq (1-x) + \delta(1-y), \tag{16}$$

$$0 \leq x \leq 1, 0 \leq y \leq 1. \tag{17}$$

Since A wants to maximize his payoff, (and since he likes more cake), he will ensure that this constraint is satisfied with equality.

$$x = \frac{1 + \delta(1-2y)}{2}, \tag{18}$$

Substituting the first constraint in the objective function, we get

$$U(y) = \frac{1 - \delta(1-2y)}{2} + \lambda(1-y) \tag{19}$$

$$\frac{\partial U}{\partial y} = \delta - \lambda > 0. \tag{20}$$

This implies that A wants to increase y up to the point that one of the non-negativity constraints in (17) are binding. The relevant constraints are $y \leq 1$, and $x \geq 0$.

Consider the case where $\delta \geq 1$. In this case, we can verify that $x \geq 0$ is the relevant constraint (since if $y = 1$, $V(x,y) > V(1-x, 1-y)$). So the solution has $x = 0$, $y = \frac{1+\delta}{2\delta}$, where the solution for y is obtained from setting $x = 0$ in (**??**).

Consider next the case where $\delta < 1$. In this case, $y \leq 1$ is the binding constraint, and we have $x = \frac{1-\delta}{2}$, obtained by setting $x = 1$ in (**??**).

Note: you can also provide a more informal argument for the solution. Intuitively, B likes Y relatively more than A, it is efficient for him to construct B's bundle so that the for any utility level that B gets from this bundle, the amount of cake Y is maximal. Second, A wants to make B indifferent between the two bundles.

b) Explain how the outcome would vary if B was to play the role of dividing the cake while A chooses.

The argument is perfectly symmetric. B would make A indifferent between the two bundles, and would design A's bundle so that x is maximized, subject to the inequality constraints, $x \leq 1$ and $y \geq 0$.

c) Is this mechanism of divide and choose a fair mechanism?

The divider has an advantage – while the chooser is indifferent between the two bundles, it is easy to verify that the divider strictly prefers his own bundle (ie. the one that the chooser does not choose).

d) Consider now an alternative divide and choose mechanism. First A divides cake X, and B chooses one of the two shares. Then B divides cake Y, and A chooses. Solve for the backwards induction outcome and compare this with of the original mechanism. Which mechanism is more efficient ?

Consider the division of cake Y first. Irrespective of what has happened on the division of cake X, A will choose the larger portion of cake Y. So it is optimal for B to set $y = 0.5$. Now in the division of cake X, for the same reason, A must set $x = 0.5$. So the division is such that both players get half of each cake. This is fair, but inefficient, since both can get a higher payoff at an allocation where A gets more of X and B gets more of Y.