C31: Game Theory. Affiliate Exam, December 2006 (Answers)

1. a) Maximizing $U(.)$ wrt to x, we get the first order condition

$$1 - 2x = 0, \tag{1}$$

which implies that player 1's best response $\hat{x}(y)$ is independent of y and equals 0.5.

Player 2's best response function is $\hat{y}(x) = x$. Solving both simultaneously, the unique NE is $(0.5, 0.5)$.

b) Solving backwards: player must choose $y = x$ whatever the value of x chosen by 1 in stage 1. Thus 1's maximization problem is to choose x to maximize:

$$1 + x - x^2 - 0.5x. \tag{2}$$

The solution is $x^* = 0.25$.

Thus the subgame perfect equilibrium is : 1 chooses 0.25; 2 chooses $y = x$ for every value of x.

c) Player 1 realizes that any change in his own action x will result in an equal change in his opponent's action in (b), He therefore takes this into account. in the simultaneous move game, a change in 1's action cannot affect 2's action.

2. a)For 1, M strictly dominates B. For 2, L strictly dominates R. After these eliminations, no strategy is strictly dominated (iteratively).

b) Since a strictly dominated strategy cannot be played in any NE, pure or mixed, we may restrict attention to $\{T, M\}$ for 1 and $\{L, C\}$ for 2.

(T, C) is a NE since neither player can do better by deviating. For example, 1 does worse by playing M, since $3 > 1$. Similarly, 2 does worse by deviating since $0 > -1$.

(M, L) is a second NE (you need to verify this)

There are no other pure strategy NE (i.e. you need to explain why (T, L) and (M, C) are not.

c) In any mixed equilibrium, players will only randomize across non-strictly dominated strategies. Let player 2 play L with prob. q, and C with prob. $1 - q$.

$$U_1(T, q) = -q + 3(1 - q). \tag{3}$$

$$U_1(M, q) = 0 + (1 - q). \tag{4}$$

Equating these payoffs one gets $q = \frac{2}{3}$.

Similarly, let 1 play T with prob. p and M with prob. $(1 - p)$. Since p must make 2 indifferent between his two actions, we can write down the payoff to actions. The solution is $p = \frac{2}{3}$.

So the Nash equilibria are the two pure strategy NE in (b) and the mixed NE set out above.

d) examples from the lectures or the book: serving in tennis or kicking penalties in football. Reporting a crime. You need to spell these out!

3 a&b)

$$U_i(b_i, v_i) = v_i \Pr(b_i \geq k_j v_j^2) - b_i \tag{5}$$
$$= v_i \Pr(\sqrt{\frac{b_i}{k_j}} \geq v_j) - b_i \tag{6}$$
$$= v_i \sqrt{\frac{b_i}{k_j}} - b_i. \tag{7}$$

where the last line follows from the fact that v_j is uniformly distributed on $[0, 1]$.

Differentiating the payoff function with respect to b_i, we get

$$\frac{1}{2} \frac{v_i}{\sqrt{k_j}} b_i^{-0.5} - 1 = 0. \tag{8}$$

This yields

$$b_i = \frac{1}{4k_j} v_i^2. \tag{9}$$

c) For a symmetric Bayes NE, we must have:

$$k_i = \frac{1}{4k_j}, \tag{10}$$

for $i = 1, 2$, where $j = 1, 2$ and $j \neq i$. That is, one must have $k_1 k_2 = \frac{1}{4}$. There are many solutions to this, one of which is $k_1 = k_2 = \frac{1}{2}$.

4. a) Suppose that $k = 1$ or 2. Then the player whose turn it is to move can

win. This implies that if $k = 3$, any move must lead to a winning position for the other player, and is therefore a losing position. This implies that if $k = 4$ or 5, the player to move can ensure that the other player is in an losing position, i.e. at $k = 3$. Thus 4 or 5 is a winning position. Now this implies that $k = 6$ is a losing position. Continuing in this fashion, one sees that $k = 15$ is a losing position.

b) The above intuition says that if n is divisible by 3, then it is a losing position, and otherwise it is a winning position, for the player who has to move. This can be proved formally by induction. Suppose that one has demonstrated that for any $k < n$, k is a winning position if it is not divisible by 3, and a losing position if it is divisible. Suppose that n is divisible by and it is i's turn to move. Any feasible move must lead to a $k' < n$ which is not divisible by 3, and therefore (by the induction hypothesis) to a winning position for i's opponent. Thus n is a losing position. Conversely, if n is not divisible by 3, then i can ensure that his opponen's position is a $k' < n$ which is divisible by 3 amd which is a losing position. Thus we have shown that n is a losing position if and only if it is divisible by 3.