Structural Econometrics: Optimal stopping

Jean-Marc Robin

Plan

1. The prototypical stationary search model
2. Adding on-the-job search and layoffs
3. Nonstationary search models

The prototypical stationary search model

Flinn, Heckman (*JoEconometrics*, 1982).

Lancaster, Chesher (*Econometrica*, 1983).

- Time is continuous and individuals live for ever.
- In a small time interval $[t, t + \Delta t]$, unemployed individuals receive a payment $b\Delta t$. Parameter b is the opportunity flow-cost of employment (including unemployment insurance, search costs, etc.).
- When unemployed, the probability of receiving an offer in $[t, t + \Delta t]$ is $\lambda_0 \Delta t$, irrespective of the time already spent waiting for an offer. Such a point process is called a Poisson process.
- Employment lasts for ever.
- The distribution of wage offers is F. The support of F is $[\underline{w}, \overline{w}] \subseteq (0, \infty)$.
- The discount rate is $\rho > 0$.

Poisson process of parameter λ_0

Define a spell as the time duration between two subsequent point realizations.

Assumptions:

- Spell durations are independent: if t_0, t_1 and t_2 are three point realizations (three subsequent job offers), and $\tau_1 = t_1 - t_0$, $\tau_2 = t_2 - t_1$, then τ_1 and τ_2 are independent.
- A spell duration has an exponential distribution with parameter λ_0:
  - the density is: $f_0(\tau) = \lambda_0 \exp(-\lambda_0 \tau)$;
  - the cdf is: $F_0(\tau) = 1 - \exp(-\lambda_0 \tau))$;
  - the expectation is: $\mathbb{E}_{\tau \sim F_0} \tau = \int_0^\infty \tau \lambda_0 e^{-\lambda_0 \tau} d\tau = \dfrac{1}{\lambda_0}$.

Important property:

At any time t, the elapsed time duration separating t from the last point realization, say τ_0, and the residual time duration separating t from the next point realization, say τ_1, are independent and have the same marginal distribution: an exponential distribution with parameter λ_0.

Employment value function

Let $W(w)$ be the value of a job offer w.

Value functions do not depend on calendar time because of the stationarity assumptions of the model (infinite horizon, Poisson processes and time-independent parameters).

Since employment spells last for ever,

$$W(w) = \int_0^{\infty} we^{-\rho t} dt = \frac{w}{\rho}.$$

Alternatively, assume discrete time with time unit Δt close to 0. Bellman's optimality principle yields:

$$\begin{aligned} W(w) &= \frac{w\Delta t}{1+\rho\Delta t} + \frac{1}{1+\rho\Delta t} W(w) \\ &\Leftrightarrow (1+\rho\Delta t) W(w) = w\Delta t + W(w) \\ &\Leftrightarrow W(w) = \frac{w}{\rho}. \end{aligned}$$

Unemployment value function

Let V_0 be the value of unemployment.

Unemployed workers accept any wage offer w such that $W(w) \geq V_0$.

Let τ denote the random residual unemployment duration. Then

$$\begin{aligned} V_0 &= \mathbb{E}_{\tau \sim F_0}\left[\int_0^\tau b e^{-\rho t} dt + e^{-\rho\tau}\mathbb{E}_{w\sim F}\max\left\{W(w), V_0\right\}\right] \\ &= \mathbb{E}_{\tau \sim F_0}\left[b\left[-\frac{e^{-\rho t}}{\rho}\right]_0^\tau + e^{-\rho\tau}\mathbb{E}_{w\sim F}\max\left\{W(w), V_0\right\}\right] \\ &= \mathbb{E}_{\tau \sim F_0}\left[\frac{b}{\rho}\left(1 - e^{-\rho\tau}\right) + e^{-\rho\tau}\mathbb{E}_{w\sim F}\max\left\{W(w), V_0\right\}\right] \\ &= \frac{b}{\rho}\left(1 - \frac{\lambda_0}{\rho + \lambda_0}\right) + \frac{\lambda_0}{\rho+\lambda_0}\mathbb{E}_{w\sim F}\max\left\{W(w), V_0\right\} \end{aligned}$$

as

$$\mathbb{E}_{\tau\sim F_0} e^{-\rho\tau} = \int_0^\infty e^{-\rho\tau}\lambda_0 e^{-\lambda_0 \tau} d\tau = \lambda_0 \int_0^\infty e^{-(\rho+\lambda_0)\tau} d\tau = \frac{\lambda_0}{\rho+\lambda_0}.$$

Hence

$$(\rho + \lambda_0)\, V_0 = b + \lambda_0 \mathbb{E}_{w\sim F}\max\left\{W(w), V_0\right\}.$$

Alternative derivation: Bellman principle

Assume discrete time with time unit Δt close to 0.

Bellman's principle yields.

$$V_0 = \frac{b\Delta t}{1+\rho\Delta t} + \frac{\lambda_0 \Delta t}{1+\rho\Delta t}\mathbb{E}_{w\sim F} \max\{W(w), V_0\} + \frac{1-\lambda_0\Delta t}{1+\rho\Delta t} V_0$$
$$\Leftrightarrow (\rho+\lambda_0) V_0 = b + \lambda_0 \mathbb{E}_{w\sim F} \max\{W(w), V_0\}.$$

Simple way of deriving these equations:

$$\underbrace{\rho V_0}_{\text{return}} = \underbrace{b}_{\text{revenue flow}} + \underbrace{\lambda_0 \mathbb{E}_{w\sim F}}_{\text{expectation}} \underbrace{[\max\{W(w), V_0\} - V_0]}_{\substack{\text{option value:} \\ \text{payoff - forlorn value}}}$$

and

$$\rho W = w$$

Unemployment value function (continued)

Lastly,

$$\begin{aligned}
\mathbb{E}_{w\sim F}\max\{W(w),V_0\} &= \mathbb{E}_{w\sim F}\max\left\{\frac{w}{\rho},V_0\right\}\\
&= V_0+\mathbb{E}_{w\sim F}\max\left\{\frac{w}{\rho}-V_0,0\right\}\\
&= V_0+\int_{\rho V_0}^{\overline{w}}\left(\frac{w}{\rho}-V_0\right)dF(w)\\
&= V_0-\left[\left(\frac{w}{\rho}-V_0\right)\overline{F}(w)\right]_{\rho V_0}^{\overline{w}}+\frac{1}{\rho}\int_{\rho V_0}^{\overline{w}}\overline{F}(w)dw\\
&= V_0+\frac{1}{\rho}\int_{\rho V_0}^{\overline{w}}\overline{F}(w)dw.
\end{aligned}$$

where $\overline{F}(w)\equiv 1-F(w)$ and where the 3rd equality is obtained by integration by part.

And we finally obtain:

$$\rho V_0=b+\frac{\lambda_0}{\rho}\int_{\rho V_0}^{\overline{w}}\overline{F}(w)dw.$$

Optimal reservation wage strategy

Since W is increasing, the optimal strategy when unemployed is to accept any wage offer w such that $w \geq \phi$, where ϕ, the reservation wage, is defined as

$$W(\phi) = V_0 \Leftrightarrow \phi = \rho V_0.$$

The optimal strategy when employed is to accept any wage offer strictly greater than the present wage contract.

The reservation wage is the solution to the equation:

$$\phi = b + \frac{\lambda_0}{\rho} \int_{\phi}^{\overline{w}} \overline{F}(w) dw.$$

The integral usually has no closed form. One should use numerical procedures to approximate the integral (quadrature, Simpson, etc.), which are available in softwares likeGauss or Matlab.

The equation can then be solved for ϕ also numerically using a Newton-type algorithm.

Data

Labor force survey usually observe workers continuously within a time interval $[t_0, t_1]$ and gather retrospective information at t_0 so that, for unemployed workers at t_0, it is possible to record:

- the elapsed unemployment duration at t_0: τ_0;
- the residual unemployment duration after t_0: τ_1 (note that $\tau_1 \leq t_1 - t_0$);
- the accepted wage w at $t_0 + \tau_1$ if the worker leaves unemployment by the end of the recording period $t_1 - t_0$.

Structural estimation

The actual unemployment duration has an exponential distribution with parameter $\lambda_0 \overline{F}(\phi)$ (instantaneous probability of receiving an offer times the probability that it be acceptable).

The Poisson property implies that τ_0 and τ_1 are independent and exponentially distributed.

The density of (τ_0, τ_1, w) is therefore equal to:

$$\ell(\tau_0, \tau_1, w) = \lambda_0 \overline{F}(\phi) \exp\left(-\lambda_0 \overline{F}(\phi)\tau_0\right) \cdot \left[\lambda_0 \overline{F}(\phi) \exp\left(-\lambda_0 \overline{F}(\phi)\tau_1\right) \cdot \frac{f(w)}{\overline{F}(\phi)}\right]^z \cdot \left[\exp\left(-\lambda_0 \overline{F}(\phi)\tau_1\right)\right]^{1-z}$$

where $z = 1$ if $\tau_1 < t_1 - t_0$, $z = 0$ if $\tau_1 = t_1 - t_0$.

Note that the distribution of wage offers is identified only conditional on $w > \phi$ (with density $\frac{f(w)}{\overline{F}(\phi)}$).

The reservation wage ϕ can be estimated by the minimal accepted job offer.

Problem with ML estimation

Parameter ϕ is a parameter of the distribution of accepted wages, i.e. $[\phi, \overline{w}]$.

Maximum Likelihood does not work in this case.

Example: sample $\{w_1, ..., w_N\}$ of N iid wages with a Pareto distribution:

$$F(w) = 1 - \left(\frac{\phi}{w}\right)^{\alpha}, \quad \alpha > 0.$$

Log-likelihood:

$$L_N(\phi, \alpha) = \sum_{i=1}^{N} \ln\left(\alpha \phi^{\alpha} w_i^{-\alpha-1}\right) = N \ln(\alpha \phi^{\alpha}) - (\alpha + 1) \sum_{i=1}^{N} \ln w_i.$$

The ML estimator of (ϕ, α) does not exist as $L_N(\phi, \alpha) \to \infty$ when $\phi \to \infty$ for any $\alpha > 0$.

Solution: estimate ϕ by $\widehat{\phi} = \min\{w_i\}$.

Superconsistent estimator, i.e. $N\left(\widehat{\phi} - \phi\right) \xrightarrow{L}$constant r.v.

Structural estimation (continued)

Estimate λ_0, parameters of F and b in **three stages**:

1. estimate reservation wage as mimimal accepted wage: $\widehat{\phi} = \min\{w_i\}$;
2. maximise log likelihood wrt λ_0 and F conditional on $\phi = \widehat{\phi}$ ($\widehat{\phi}$ is like a constant as it is super-consistent);
3. estimate b as

$$\widehat{b} = \widehat{\phi} - \frac{\widehat{\lambda}_0}{\rho} \int_{\widehat{\phi}}^{\overline{w}} \overline{\widehat{F}}(w)dw.$$

Adding on-the-job search and layoffs

We make the same assumptions as in the basic search model except that we now assume in addition that:

- When employed, job offers accrue at constant rate λ_1. (It is likely that $\lambda_1 < \lambda_0$.)
- Job are exogenously destroyed at constant rate δ.

Employees at wage w accept an alternative job paid x if only $W(x) > W(w)$.

Value functions

The unemployment value function proceeds from the same definition as before:

$$\rho V_0 = b + \lambda_0 \mathbb{E}_{w\sim F} \max\{W(w) - V_0, 0\}.$$

Deriving the employment value function $W(w)$ is slightly more complicated.

Using the Bellman principle we have:

$$\begin{aligned} W(w) &= \frac{w\Delta t}{1+\rho\Delta t} + \frac{\lambda_1 \Delta t}{1+\rho\Delta t}\mathbb{E}_{x\sim F} \max\{W(x), W(w)\} + \frac{\delta\Delta t}{1+\rho\Delta t}V_0 \\ &\quad + \frac{1-\delta\Delta t - \lambda_1\Delta t}{1+\rho\Delta t}W(w) \\ \Leftrightarrow (\rho + \lambda_1 + \delta)\, W(w) &= w + \lambda_1 \mathbb{E}_{x\sim F} \max\{W(x), W(w)\} + \delta V_0 \\ \Leftrightarrow \rho W(w) &= w + \lambda_1 \mathbb{E}_{x\sim F} \max\{W(x) - W(w), 0\} + \delta\,(V_0 - W(w)). \end{aligned}$$

W is continuous

For all real z, x and y,

$$|\max\{z,x\} - \max\{z,y\}| \leq |x-y|\,.$$

(Show that by trying all three cases: $z < x < y$, $x < z < y$ and $x < y < z$.)

Moreover, for all couple of rv X, Y, $|\mathbb{E}X - \mathbb{E}Y| \leq \mathbb{E}\,|X-Y|\,.$

Hence

$$\begin{aligned}
&|\mathbb{E}_w \max\{W(w), W(x)\} - \mathbb{E}_w \max\{W(w), W(y)\}| \\
&\qquad \leq \mathbb{E}_w \,|\max\{W(w), W(x)\} - \max\{W(w), W(y)\}| \\
&\qquad\qquad \leq \mathbb{E}_w \,|W(x) - W(y)| = |W(x) - W(y)|
\end{aligned}$$

and since

$$(\rho + \lambda_1 + \delta)\, W(w) = w + \lambda_1 \mathbb{E}_{x\sim F} \max\{W(x), W(w)\} + \delta V_0,$$

it follows that:

$$(\rho + \delta)\,|W(x) - W(y)| \leq |x-y|\,,$$

so W is Lipschitz-continuous.

W is increasing

For all k, x and y

$$\max\{k, W(x)\} - \max\{k, W(y)\} \geq \min\{W(x) - W(y), 0\}.$$

Suppose now that $x > y$. Then,

$$(\rho + \lambda_1 + \delta)\,[W(x) - W(y)] \geq x - y + \lambda_1 \min\{W(x) - W(y), 0\}.$$

And if $W(x) - W(y) \leq 0$, then

$$(\rho + \delta)\,[W(x) - W(y)] \geq x - y > 0,$$

yielding a contradiction.

Hence $W(x) > W(y)$.

Optimal reservation wage strategy

Since W is increasing, the optimal strategy when unemployed is to accept any wage offer w such that $w \geq \phi$, where ϕ, the reservation wage, is defined as

$$W(\phi) = V_0 \Leftrightarrow \phi = W^{-1}(V_0).$$

The optimal strategy when employed is to accept any wage offer strictly greater than the present wage contract.

The reservation wage is the solution to the equation:

$$\phi = b + (\lambda_0 - \lambda_1) \int_{\phi}^{\overline{w}} \frac{\overline{F}(w)dw}{\rho + \delta + \lambda_1 \overline{F}(w)}.$$

Proof of reservation price formula

One has

$$\begin{aligned}
\rho V_0 &= b + \lambda_0 \mathbb{E}_{w \sim F} \max \{W(w) - V_0, 0\} \\
&= b + \lambda_0 \int_{\underline{w}}^{\overline{w}} \max \{W(w) - V_0, 0\} \, dF(w) \\
&= b + \lambda_0 \int_{\phi}^{\overline{w}} [W(w) - V_0] \, dF(w) \\
&= b + \lambda_0 \left[(W(w) - V_0) \, \overline{F}(w) \right]_{\phi}^{\overline{w}} + \lambda_0 \int_{\phi}^{\overline{w}} \overline{F}(w) dW(w) \\
&= b + \lambda_0 \int_{\phi}^{\overline{w}} \overline{F}(w) dW(w)
\end{aligned}$$

after integrating by part to obtain the 3rd equality.

Integration by part also yield the following result:

$$\begin{aligned}
\rho W(w) &= w + \lambda_1 \mathbb{E}_{x\sim F} \max\left\{W(x) - W(w), 0\right\} + \delta\left(V_0 - W(w)\right) \\
&= w + \lambda_1 \int_w^{\overline{w}} \left[W(x) - W(w)\right] dF(x) + \delta\left[V_0 - W(w)\right] \\
&= w + \lambda_1 \int_w^{\overline{w}} \overline{F}(x) dW(x) + \delta\left[V_0 - W(w)\right].
\end{aligned}$$

Differentiating this latter equation with respect to w yields:

$$\begin{aligned}
\rho W'(w) &= 1 - \lambda_1 \overline{F}(w) W'(w) - \delta W'(w) \\
&\Leftrightarrow W'(w) = \frac{1}{\rho + \delta + \lambda_1 \overline{F}(w)} \\
&\Rightarrow W(w) = \underbrace{\frac{\overline{w} + \delta V_0}{\rho + \delta}}_{= W(\overline{w})} - \int_w^{\overline{w}} \frac{dx}{\rho + \delta + \lambda_1 \overline{F}(x)}
\end{aligned}$$

as

$$\begin{aligned}
\rho W(\overline{w}) &= \overline{w} + \lambda_1 \mathbb{E}_{x\sim F} \max\left\{W(x) - W(\overline{w}), 0\right\} + \delta\left(V_0 - W(\overline{w})\right) \\
&= \overline{w} + \delta\left(V_0 - W(\overline{w})\right).
\end{aligned}$$

Lastly, reporting the value for $W'(x)$ into the equation for V_0 yields

$$\rho V_0 = b + \lambda_0 \int_{\phi}^{\overline{w}} \overline{F}(w) dW(w) = b + \lambda_0 \int_{\phi}^{\overline{w}} \frac{\overline{F}(w)dw}{\rho + \delta + \lambda_1 \overline{F}(w)}.$$

Alternatively, by definition,

$$\begin{aligned}
\rho V_0 &= \rho W(\phi) \\
&= \phi + \lambda_1 \mathbb{E}_{x \sim F} \max \{W(x) - W(\phi), 0\} + \delta (V_0 - W(\phi)) \\
&= \phi + \lambda_1 \mathbb{E}_{x \sim F} \max \{W(x) - W(\phi), 0\} \\
&= \phi + \lambda_1 \int_{\phi}^{\overline{w}} \overline{F}(x) dW(x).
\end{aligned}$$

Taking the term-by-term difference of these latter two equations then achieves to show that:

$$\phi = b + (\lambda_0 - \lambda_1) \int_{\phi}^{\overline{w}} \frac{\overline{F}(w)dw}{\rho + \delta + \lambda_1 \overline{F}(w)}.$$

Nonstationary search models

Van den Berg (*REStud*, 1990).

We here consider another extension of the prototypical search model: b now depends on unemployment duration.

For simplicity, we rule out on-the-job search and layoffs, so that the employment value continues to be $W(w) = \frac{w}{\rho}$.

Let $V_0(t)$ denote the value of unemployment for unemployment duration t.

The Bellman principle allows to write:

$$V_0(t) = \frac{b(t)\Delta t}{1+\rho\Delta t} + \frac{\lambda_0 \Delta t}{1+\rho\Delta t}\mathbb{E}_{w\sim F} \max\left\{W(w), V_0(t+\Delta t)\right\} + \frac{1-\lambda_0\Delta t}{1+\rho\Delta t}V_0(t+\Delta t).$$

Hence

$$\rho V_0(t) = b(t) + \lambda_0 \mathbb{E}_{w\sim F} \max\left\{W(w) - V_0(t+\Delta t), 0\right\} + \frac{V_0(t+\Delta t) - V_0(t)}{\Delta t}.$$

Letting $\Delta t \to 0$, we obtain:

$$\begin{aligned}
\rho V_0(t) &= b(t) + \lambda_0 \mathbb{E}_{w\sim F} \max\left\{W(w) - V_0(t), 0\right\} + \frac{dV_0(t)}{dt} \\
&= b(t) + \lambda_0 \int_{\rho V_0(t)}^{\overline{w}} \left(\frac{w}{\rho} - V_0(t)\right) dF(w) + \frac{dV_0(t)}{dt} \\
&= b(t) + \frac{\lambda_0}{\rho} \int_{\rho V_0(t)}^{\overline{w}} \overline{F}(w) dw + \frac{dV_0(t)}{dt}.
\end{aligned}$$

This is an integral-differential equation. Differentiating, once more, we obtain a non-linear second-order differential equation:

$$\left(\rho + \frac{\lambda_0}{\rho}\overline{F}\left[\rho V_0(t)\right]\right)\frac{dV_0(t)}{dt} = b'(t) + \frac{d^2 V_0(t)}{dt^2}.$$

This equation is solved backward, assuming that $b(t) = b_0$ for all $t \geq t_0$. The problem is stationary after t_0, whereby

$$\rho V_0(t_0) = b_0 + \frac{\lambda_0}{\rho} \int_{\rho V_0(t_0)}^{\overline{w}} \overline{F}(w) dw.$$

$V_0(t_0)$ then provides a terminal condition for the differential equation.

Estimation

Maximum likelihood.

The computation of reservation wage functions requires numerical algoritms. Matlab proposes numerous ODE solvers.

Results: see Van den Berg's paper.

See also Meyer (*Econometrica*, 1990) for a good descriptive-but-inspired-by-the-search-model econometric analysis of unemployment insurance on unemployment duration.