# Structural Econometrics:
# Consumption Dynamics

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**Plan**

1. The life-cycle model – Hall (JPE, 1978)
2. Estimation of Hall's lifecycle model – Hall and Mishkin (Econometrica, 1982)
3. Liquidity constraints – Zeldes (JPE, 1986)

# The life-cycle model

Hall (*JPE*, 1978).

- Discrete time.
- At the beginning of each period $t$, a household decides about consumption for period $t$ given disposable income.
- Notations:

| | | | |
|---|---|---|---|
| $a_t$ | savings | $R_t$ | disposable income |
| $c_t$ | consumption | $H$ | life duration |
| $y_t$ | income flow | $\rho$ | discount rate |
| $r_t$ | interest rate | $U_t$ | VNM utility function |

- **Budget constraint:** consumption + savings = disposable income

$$\begin{aligned} c_t + a_t &= R_t \\ &= y_t + (1 + r_t)a_{t-1}. \end{aligned}$$

- **State variables:** everything that is known by household at decision time $t$. Includes current disposable income $R_t$ and present and past income flows and interest rates: $\mathbf{y}_t = (y_t, y_{t-1}, ...)$ and $\mathbf{r}_t = (r_t, r_{t-1}, ...)$. In the sequel, we omit the reference to exogenous realisations $\mathbf{y}_t$ and $\mathbf{r}_t$ (shocks).

- **Consumption strategy:** a function $c_t(R_t; \mathbf{y}_t, \mathbf{r}_t)$ of state variables.

## Value function and Bellman equation

- **Value function** = Expected sum of future utility flows for optimal consumption strategy:

$$V_t(R_t) = \max_{c_t(\cdot), c_{t+1}(\cdot), \ldots} \mathbb{E}_t \left( \sum_{i=0}^{H-t} \rho^i U_{t+i}(c_{t+i}) \right),$$

where $\mathbb{E}_t(\cdot) = \mathbb{E}(\cdot|\mathbf{y}_t, \mathbf{r}_t)$ is expectation with respect to future shocks given present and past shocks.

- **Terminal condition** = no bequest:

$$R_{H+1} = (1 + r_{H+1})a_H = 0.$$

- **Bellman equation**: given that tomorrow I will behave optimally, what does that mean to behave optimally today?

$$\begin{aligned} V_t(R_t) &= \max_c U_t(c) + \rho \mathbb{E}_t V_{t+1}(R_{t+1}) \\ \text{s.t. } R_{t+1} &= y_{t+1} + (1 + r_{t+1})(R_t - c). \end{aligned}$$

## Euler equation

- **Optimisation problem:**

$$V_t(R_t) = \max_c U_t(c) + \rho \mathbb{E}_t V_{t+1}\left(y_{t+1} + (1 + r_{t+1})(R_t - c)\right).$$

- **First Order Condition**: optimal consumption $c_t\left(R_t\right)$ solves

$$\begin{aligned} U_t'(c_t) &= \rho \mathbb{E}_t (1 + r_{t+1}) V_{t+1}'\left(y_{t+1} + (1 + r_{t+1})(R_t - c_t)\right) \\ &= \rho \mathbb{E}_t (1 + r_{t+1}) V_{t+1}'\left(R_{t+1}\right) \end{aligned} \tag{1}$$

- Apply the **envelope theorem**:

$$\begin{aligned} V_t'(R_t) &= \rho \mathbb{E}_t (1 + r_{t+1}) V_{t+1}'\left[y_{t+1} + (1 + r_{t+1})(R_t - c_t)\right] \\ &= \rho \mathbb{E}_t (1 + r_{t+1}) V_{t+1}'\left(R_{t+1}\right) \\ &= U_t'(c_t). \end{aligned} \tag{2}$$

- **Euler equation:** use (2) to replace $V_{t+1}'\left(R_{t+1}\right)$ by $U_{t+1}'\left(c_{t+1}\right)$ in (1). Then

$$U_t'(c_t) = \rho \mathbb{E}_t\left[(1 + r_{t+1}) U_{t+1}'\left(c_{t+1}\right)\right].$$

Sort of random walk equation for consumption: best predictor of tomorrow's consumption is today's consumption.

Hypothesis usually not satisfied by macro data: regress $c_t - c_{t-1}$ on $c_{t-1}$ and $y_t$; significant effect of $y_t$.

# Estimation of Hall's model using panel data

Hall et Mishkin (*Ecta*, 1982).

Construction of an econometric model generally requires additional specification assumptions:

- Quadratic utility function: $U_t(c_t) = -(c_t^* - c_t)^2, \quad c_t \leq c_t^*$.
- Constant interest rate: $r_t = r$.
- Discount rate: $\rho = (1+r)^{-1}$.
- Income process: $y_t = y_t^D + y_t^P + y_t^T$, where
  - $y_t^D$ is a **deterministic trend**,
  - $y_t^P = y_{t-1}^P + \varepsilon_t = y_{t_0}^P + \varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_{t_0+1}$ is **permanent income** ($t_0$ beginning of lifecycle; innovation $\varepsilon_t$ is a white noise),
  - $y_t^T = \Phi(L)\eta_t = \phi_0\eta_t + \phi_1\eta_{t-1} + ... + \phi_q\eta_{t-q}$ is **transitory income** ($\eta_t$ is a white noise):
    * For $\eta_t$ to be an **innovation** (i.e. $\frac{1}{\Phi(L)}y_t^T$ only depends on the past of $y_t^T$), the roots of the characteristic polynomial are strictly outside the unit circle.
    * Assume $\phi_0 = 1$ for normalisation, as $\text{Var}(\eta_t)$ and $\Phi$ are not separately identifiable.

# Euler equations

- Euler equations become:

$$U_t'(c_t) = \mathbb{E}_t\left[U_{t+1}'\left(c_{t+1}\right)\right]$$

or

$$\mathbb{E}_t c_{t+1} - c_{t+1}^* = c_t - c_t^*$$

as

$$U_t'(c_t) = 2\left(c_t^* - c_t\right).$$

- And the Law of iterated expectations implies

$$\begin{aligned} \mathbb{E}_t c_{t+i} - c_{t+i}^* &= \mathbb{E}_t\left(\mathbb{E}_{t+1}\left(\ldots\left(\mathbb{E}_{t+i-1}\left(c_{t+i} - c_{t+i}^*\right)\right)\right)\right) \\ &= c_t - c_t^*. \end{aligned}$$

# Analytical solution

Apply Euler equations to the sequence of budget constraints.

- Intertemporal budget constraint:

$$\left\{\begin{array}{c} R_{t+1} = y_{t+1} + (1+r)(R_t - c_t) \\ R_{t+2} = y_{t+2} + (1+r)(R_{t+1} - c_{t+1}) \\ \vdots \\ R_H = y_H + (1+r)(R_H - c_H) \\ R_{H+1} = 0 = (1+r)(R_H - c_H) \end{array}\right\} \Longrightarrow \left\{\begin{array}{c} R_t = y_t + (1+r)a_{t-1} \\ \dfrac{R_{t+1}}{1+r} = \dfrac{y_{t+1}}{1+r} + R_t - c_t \\ \dfrac{R_{t+2}}{(1+r)^2} = \dfrac{y_{t+2}}{(1+r)^2} + \dfrac{R_{t+1} - c_{t+1}}{1+r} \\ \vdots \\ \dfrac{R_H}{(1+r)^{H-t}} = \dfrac{y_H}{(1+r)^{H-t}} + \dfrac{R_{H-1} - c_{H-1}}{1+r} \\ \dfrac{R_{H+1}}{(1+r)^{H-t+1}} = 0 = \dfrac{R_H - c_H}{(1+r)^{H-t}} \end{array}\right\}$$

Summing up,

$$0 = \sum_{i=0}^{H-t} \frac{y_{t+i}}{(1+r)^i} + (1+r)a_{t-1} - \sum_{i=0}^{H-t} \frac{c_{t+i}}{(1+r)^i}.$$

- Take expectation:

$$\begin{aligned}
0 &= \sum_{i=0}^{H-t} \frac{\mathbb{E}_t y_{t+i}}{(1+r)^i} + (1+r)a_{t-1} - \sum_{i=0}^{H-t} \frac{\mathbb{E}_t c_{t+i}}{(1+r)^i} \\
&= \sum_{i=0}^{H-t} \frac{\mathbb{E}_t y_{t+i}}{(1+r)^i} + (1+r)a_{t-1} - \sum_{i=0}^{H-t} \frac{c^*_{t+i} + c_t - c^*_t}{(1+r)^i}
\end{aligned}$$

  using Euler equations.

- Hence the **consumers demands**

$$\boxed{c_t = c^*_t + \gamma_t^{-1}\left[(1+r)a_{t-1} + Y_t - C^*_t\right]}$$

  with

$$\gamma_t = \sum_{i=0}^{H-t} \frac{1}{(1+r)^i}, \quad Y_t = \sum_{i=0}^{H-t} \frac{\mathbb{E}_t y_{t+i}}{(1+r)^i}, \quad C^*_t = \sum_{i=0}^{H-t} \frac{c^*_{t+i}}{(1+r)^i}.$$

- **Consumption smoothing**: expected wealth $(1+r)a_{t-1} + Y_t - C^*_t$ is spread evenly over the lifecycle.

## Expected wealth

The income process implies

$$
\begin{aligned}
\mathbb{E}_t y_t &= y_t \\
&= y_t^D + y_t^P + y_t^T \\
&= y_t^D + y_{t-1}^P + \varepsilon_t + \eta_t + \phi_1 \eta_{t-1} + \ldots + \phi_q \eta_{t-q}, \\
\mathbb{E}_t y_{t+1} &= y_{t+1}^D + \mathbb{E}_t y_{t+1}^P + \mathbb{E}_t y_{t+1}^T \\
&= y_{t+1}^D + y_{t-1}^P + \varepsilon_t + \phi_1 \eta_t + \ldots + \phi_q \eta_{t+1-q}, \\
\mathbb{E}_t y_{t+i} &= y_{t+i}^D + \mathbb{E}_t y_{t+i}^P + \mathbb{E}_t y_{t+i}^T \\
&= \left| \begin{array}{ll} y_{t+i}^D + y_{t-1}^P + \varepsilon_t + \phi_i \eta_t + \ldots + \phi_q \eta_{t+i-q} & (i < q), \\ y_{t+i}^D + y_{t-1}^P + \varepsilon_t + \phi_q \eta_t & (i = q), \\ y_{t+i}^D + y_{t-1}^P + \varepsilon_t & (i > q). \end{array} \right.
\end{aligned}
$$

Hence **process** $Y_t$ can be represented as

$$
Y_t = \sum_{i=0}^{H-t} \frac{y_{t+i}^D}{(1+r)^i} + \gamma_t y_{t-1}^P + \gamma_t \varepsilon_t + \sum_{j=0}^{q} \left( \sum_{i=0}^{\min\{H-t, q-j\}} \frac{\phi_{i+j}}{(1+r)^i} \right) \eta_{t-j}.
$$

## Consumption innovation

- Euler equations show that the consumption process is a random walk with a deterministic drift, that is $c_t - c_t^* - \left(c_{t-1} - c_{t-1}^*\right)$ is a martingale difference sequence.

(Note that a process $(u_t)$ is called a martingale difference sequence if $\mathbb{E}_{t-1} u_t = 0$.)

- Expectation of $c_t$ given time-$(t-1)$ information is

$$\begin{aligned} \mathbb{E}_{t-1} c_t &= \mathbb{E}_{t-1}\left\{c_t^* + \gamma_t^{-1}\left[(1+r)a_{t-1} + Y_t - C_t^*\right]\right\} \\ &= c_t^* + \gamma_t^{-1}\left[(1+r)a_{t-1} + \mathbb{E}_{t-1} Y_t - C_t^*\right]. \end{aligned}$$

- Hence

$$c_t - \mathbb{E}_{t-1} c_t = \gamma_t^{-1}\left(Y_t - \mathbb{E}_{t-1} Y_t\right) = \gamma_t^{-1}\left(\gamma_t \varepsilon_t + \sum_{i=0}^{\min\{H-t,q\}} \frac{\phi_i}{(1+r)^i} \eta_t\right)$$

or

$$\boxed{c_t - c_t^* = c_{t-1} - c_{t-1}^* + \varepsilon_t + \beta_t \eta_t}$$

with

$$\beta_t = \gamma_t^{-1} \sum_{i=0}^{\min\{H-t,q\}} \frac{\phi_i}{(1+r)^i}.$$

# Data

Panel Study of Income Dynamics (PSID) 1969-75

Consommation = food.

- "How much do you spend on food in an average week ?"
- Question asked in March.

Income is last calendar year's income.

# Econometric model

(Simpler version of Hall and Mishkin's)

- $H = \infty$, so $\gamma_t = \gamma = 1 + \frac{1}{r}$ and $\beta_t = \beta$ constant.
- Transitory income is MA(1): $y_t^T = \eta_t + \phi\eta_{t-1}$.
- To take into account that food is not total consumption and the different timing of income and consumption measures, the econometric model assumes:

$$\begin{cases} \Delta c_t = \Delta c_t^* + \alpha[\delta(\varepsilon_{t+1} + \beta\eta_{t+1}) + (1-\delta)(\varepsilon_t + \beta\eta_t)], \\ \Delta y_t = \Delta y_t^D + \varepsilon_t + \eta_t - (1-\phi)\eta_{t-1} - \phi\eta_{t-2}. \end{cases}$$

  where

  - $\alpha$ is the share of total weath spent on food and
  - $\delta$ models the advance of information on future income.

- $\Delta c_t^*$ and $\Delta y_t^D$ are linear functions of age, age squared, demographics and log price changes.

**Remarks:**

- Hall and Mishkin add another MA(1) component to $\Delta c_t$ to account for measurement errors.
- Transitory income is a MA(2) process.
- Note that there is no individual specific effect although the data are panel data.

# Estimation

Use OLS to estimate $\Delta c_t^*$ and $\Delta y_t^D$ functions of exogenous variables.

Then maximise likelihood of residuals to estimate $\alpha, \delta, \beta, \phi$ and variances of $\varepsilon_t$ and $\eta_t$.

Results:

- $\alpha = 0.11$ (mean propensity to consume on raw data: 0.19.)
- $\beta = 0.29$ (slightly above the theoretical value that can be computed using standard interest rate variables).
- $\delta = 0.25$ (if consumption observed in March; should depend for 3/4 of past income and for 1/4 of current year income; bingo!)

## Excess sensitivity to transitory income

- Model predicts covariances well except for $\text{Cov}(\Delta c_t, \Delta y_{t-1})$ which is theoretically nil as

$$\Delta c_t = \varepsilon_t + \beta \eta_t \quad \text{and} \quad \Delta y_{t-1} = \varepsilon_{t-1} + \eta_{t-1} + (\phi - 1)\,\eta_{t-2} - \phi \eta_{t-3}$$

- Yet, regressing $\Delta c_t$ on $\Delta y_{t-1}$ yields:

$$\Delta c_t = \underset{(6.16)}{-4.95} - \underset{(.002)}{0.010} \Delta y_{t-1}.$$

- Negative correlation can be explained if household population is heterogenous:
  - a fraction $\mu$ of households track income, i.e. $\Delta c_t = \alpha \Delta y_t$.
  - a fraction $1 - \mu$ behaves as in the theory.

  Then for constrained households, $\text{Cov}(\Delta c_t, \Delta y_{t-1}) = -\alpha\,(\phi - 1)^2\, \sigma_\eta^2 < 0$.

- Hall and Mishkin estimate $\mu = .20$.
- Note that this ad hoc change in the model does not solve the problem completely. Predicted $\text{Cov}(\Delta c_t, \Delta y_{t-1})$ is $-0.032$ instead of $-0.077$.

# Liquidity constraints

Zeldes (*JPE*, 1989)

Structural model of households constrained by credit rationing.

- **Liquidity constraint**: Households cannot borrow more than a certain amount:

$$a_t \geq \underline{a}_t \text{ pour tout } t,$$

where $\underline{a}_t < 0$.

- Implies the following constraint on consumption:

$$a_t = R_t - c_t \geq \underline{a}_t$$

or

$$c_t \leq \overline{c}_t = R_t - \underline{a}_t$$

where $R_t = y_t + (1 + r_t)a_{t-1}$.

## Optimal consumption at $t$ given optimal behaviour at $t+1$

- Bellman equation:

$$V_t(R_t) = \max_c U_t(c) + \rho \mathbb{E}_t V_{t+1}(R_{t+1}) \quad \text{s.t.} \begin{cases} R_{t+1} = y_{t+1} + (1+r_{t+1})(R_t - c) \\ c \leq \overline{c}_t \end{cases}$$

- Kuhn and Tucker conditions:

$$\begin{aligned} U_t'(c_t) - \rho \mathbb{E}_t \left[ (1+r_{t+1}) V_{t+1}' \left( y_{t+1} + (1+r_{t+1})(R_t - c_t) \right) \right] - \widetilde{\mu}_t &= 0, \\ \widetilde{\mu}_t \left( c_t - \overline{c}_t \right) &= 0. \end{aligned}$$

where $\widetilde{\mu}_t \geq 0$ is Kuhn and Tucker's constant.

- Two cases:
  - Corner solution: $\widetilde{\mu}_t > 0$ and $c_t = \overline{c}_t$,
  - Interior solution: $\widetilde{\mu}_t = 0$ and $c_t = \widetilde{c}_t \leq \overline{c}_t$, where $\widetilde{c}_t$ solves

$$U_t'(\widetilde{c}_t) - \rho \mathbb{E}_t \left[ (1+r_{t+1}) V_{t+1}' \left( y_t + (1+r_t) a_{t-1} - \widetilde{c}_t \right) \right] = 0.$$

  - Corner solution optimal iff $\widetilde{c}_t > \overline{c}_t = R_t - \underline{a}_t$.

# Euler equation

- Value function:

$$\begin{aligned} V_t(R_t) &= U_t(c_t) + \rho \mathbb{E}_t V_{t+1}\left(y_{t+1} + (1+r_{t+1})(R_t - c_t)\right) \\ &= \left| \begin{array}{ll} U_t(R_t - \underline{a}_t) + \rho \mathbb{E}_t V_{t+1}\left(y_{t+1} + (1+r_{t+1})\underline{a}_t\right) & \text{if } \widetilde{c}_t > \overline{c}_t = R_t - \underline{a}_t, \\ U_t(\widetilde{c}_t) + \rho \mathbb{E}_t V_{t+1}\left(y_{t+1} + (1+r_{t+1})(R_t - \widetilde{c}_t)\right) & \text{if } \widetilde{c}_t \leq R_t - \underline{a}_t. \end{array} \right. \end{aligned}$$

- **Envelope theorem**:

$$V_t'(R_t) = \left| \begin{array}{ll} U_t'(R_t - \underline{a}_t) = U_t'(\overline{c}_t) & \text{if } \widetilde{c}_t > \overline{c}_t = R_t - \underline{a}_t, \\ \rho \mathbb{E}_t\left[(1+r_{t+1})V_{t+1}'\left(y_{t+1} + (1+r_{t+1})(R_t - \widetilde{c}_t)\right)\right] = U_t'(\widetilde{c}_t) & \text{if } \widetilde{c}_t \leq R_t - \underline{a}_t. \end{array} \right.$$

  Hence: $V_t'(R_t) = U_t'(c_t)$.

- Use **Kuhn and Tucker conditions** again to derive Euler equation:

$$\begin{aligned} U_t'(c_t) &= \rho \mathbb{E}_t\left[(1+r_{t+1})V_{t+1}'(R_{t+1})\right] + \widetilde{\mu}_t \\ &\Updownarrow \\ U_t'(c_t) &= \rho \mathbb{E}_t\left[(1+r_{t+1})U_{t+1}'(c_{t+1})\right] + \widetilde{\mu}_t \end{aligned}$$

  or

$$\mathbb{E}_t\left[\frac{\rho(1+r_{t+1})U_{t+1}'(c_{t+1})}{U_t'(c_t)}\right] = 1 - \frac{\widetilde{\mu}_t}{U_t'(c_t)} \equiv \frac{1}{1+\mu_t} \qquad (\mu_t \geq 0).$$

## Data and econometric specification

- PSID

Consumption = Food

- Assumption: utility function additively separable between food and other other consumptions.

Implies that previous theory applies exactly the same to food consumption.

- Econometric specification: household $i$, period $t$,

$$U_{it}(c_{it}) = \frac{c_{it}^{1-\alpha}}{1-\alpha} \exp \theta_{it}$$

where $\theta_{it}$ is a linear index of exogenous variables:

$$\theta_{it} = b_0 Age_{it} + b_1 Age_{it}^2 + b_2 \ln(n_{it}) + u_i + u_t + u_{it},$$

where $n_{it}$ is household size.

# Econometric model of consumption dynamics

- For non constrained households, Euler equations take the form:

$$\frac{c_{it+1}^{-\alpha} \exp\left[\theta_{it+1} - \theta_{it}\right] \rho(1 + r_{it+1})(1 + \mu_{it})}{c_{it}^{-\alpha}} = 1 + \varepsilon_{it+1}$$

  where $\varepsilon_{it+1}$ is a martingale difference sequence ($\mathbb{E}_t\varepsilon_{it+1} = 0$).

- Take logs:

$$\Delta \ln c_{it+1} = \frac{1}{\alpha}\left[\Delta\theta_{it+1} + \ln\rho + \ln(1 + r_{it+1}) + \ln(1 + \mu_{it}) - \ln(1 + \varepsilon_{it+1})\right].$$

- Second order approximation:

$$\ln(1 + \varepsilon_{it+1}) = \varepsilon_{it+1} - \frac{1}{2}\varepsilon_{it+1}^2 \Rightarrow \mathbb{E}_t \ln(1 + \varepsilon_{it+1}) \simeq -\frac{1}{2}\mathrm{Var}_t\varepsilon_{it+1}.$$

- Final specification:

$$\Delta \ln c_{it+1} = \kappa_t + \tfrac{1}{\alpha}\ln(1 + r_{it+1}) + \tfrac{2b_1}{\alpha}Age_{it} + \tfrac{b_2}{\alpha}\Delta \ln n_{it+1} + \frac{\ln(1 + \mu_{it})}{\alpha} + \eta_{it+1}$$

  where

$$\begin{aligned}
\kappa_t &= \frac{1}{\alpha}\left[\ln\rho + b_0 + b_1 + \Delta u_{t+1} + \frac{1}{2}\mathrm{Var}_t\varepsilon_{it+1}\right] \\
\eta_{it+1} &= \frac{1}{\alpha}\left[\Delta u_{it+1} - \ln(1 + \varepsilon_{it+1}) + \mathbb{E}_t \ln(1 + \varepsilon_{it+1})\right] \sim MA(1) \text{ process.}
\end{aligned}$$

# Estimation

- $r_{it+1}$ possibly endogenous.
  - **Assumption**: $\ln y_{it}, a_{it-1}/y_{it}$, where $a_{it-1}$ is measured by household assets are valid instruments.
  - **Estimation**: 2SLS on $\Delta \ln c_{it} - \overline{(\Delta \ln c_{it})}_i$ with time dummies.
- Assume that most of the individual fixed effect results from $\frac{\ln(1+\mu_{it})}{\alpha}$ and
  - estimate $\overline{\left(\frac{\ln(1+\mu_{it})}{\alpha}\right)}_i$ as mean residuals

$$\overline{(\Delta \ln c_{it})}_i - \overline{\left(\kappa_t + \frac{1}{\alpha}\ln(1+r_{it+1}) + \frac{2b_1}{\alpha}Age_{it} + \frac{b_2}{\alpha}\Delta \ln n_{it+1} + \frac{\ln(1+\mu_{it})}{\alpha}\right)}_i$$

## Test of liquidity constraints

Partition sample into poor and rich household, according to last period's assets $a_{i,t-1}$.

- **Test 1**: Estimate model on both subsamples incorporating $\ln y_{it}$ in the equation but without $\ln(1+\mu_{it})$.

  **Result**: the coefficient of $\ln y_{it}$ comes out significantly only for the poor.

- **Test 2** :

  1. Estimate the model on the sample of rich households.
  2. Compute $\overline{\left(\frac{\ln(1+\mu_{it})}{\alpha}\right)_i}$ from poor subsample using parameters estimated at stage 1.

  **Résultat** : mean of $\overline{\left(\frac{\ln(1+\mu_{it})}{\alpha}\right)_i}$ positive (.017) but not significant.

- **Test 3** : Regress estimated residuals $(\ln(1+\mu_{it})+\eta_{it})$ on $\ln y_{it}$.

  **Result** : negative coefficient, not significant.

Note that different ways of partitioning the sampe gave similar results.