Characterization of the extreme core allocations of the assignment game

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Version: January 30, 2002

[1]This work has been supported by the University of Barcelona (Divisió II) and by Spanish Research grant DGICYT Project PB98-0867

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A characterization of the extreme core allocations of the assignment game is given in terms of the reduced marginal worth vectors. For each ordering in the player set, a payoff vector is defined where each player receives his or her marginal contribution to a certain reduced game played by his or her predecessors. This set of reduced marginal worth vectors, which for convex games coincide with the usual marginal worth vectors, is proved to be the set of extreme points of the core of the assignment game. Therefore, although assignment games are hardly ever convex, the same characterization of extreme core allocations is valid for convex games.

Key Words: Extreme core allocations, assignment game, reduced marginal worth vectors, convex games.

Journal of Economic Literature: C71, C78

1. INTRODUCTION

In 1971, Shapley and Shubik studied a two–sided market where the agents are buyers and sellers and one good is present in indivisible units. Each seller owns a unit of the indivisible good and each buyer needs exactly one unit. Differentiation on the units is allowed and therefore a buyer might place different valuations on the units of different sellers. This model is known as the assignment game. Apart from the original paper, the reader is referred to Shubik (1985) and Roth and Sotomayor (1990) for a general presentation.

Under the assumption that side payments among agents are allowed, and identifying utility with money, Shapley and Shubik proved that the core of the assignment game (that is to say the set of efficient outcomes that no coalition can improve upon) is always nonempty and can be identified with the set of competitive equilibria of the market.

Recently, further research has been made into the classical assignment game of Shapley and Shubik, and this is the framework of our paper. Hamers *et al.* (2002) prove that every extreme core allocation of an assignment game is a marginal worth vector.

The present paper is mainly devoted to a generalization of marginal worth vectors, introducing what we call reduced marginal worth vectors and proving that they coincide with the set of extreme core allocations of the assignment game. We are looking for a global characterization of the extreme core points of the assignment game, as Shapley did for convex games, and this is the difference with Hamers' work.

In his seminal paper devoted to convex games, Shapley (1971) introduced the marginal worth vectors for the general framework of cooperative games with transferable utility. In each one of these vectors, which are all efficient, each player is paid his marginal contribution to his set of predecessors according to a fixed permutation over the player set. It is well known that the set of marginal worth vectors coincides with the set of extreme points of the core only when the game is convex. As assignment games are not always convex, the above coincidence does not hold in general.

Our reduced marginal worth vectors are inspired in the classical marginal worth vectors with the difference that, for a fixed permutation on the player set, a reduction of the game is performed before each player is paid her marginal contribution to her set of predecessors. Moreover, for convex games, reduced marginal worth vectors will coincide with the marginal worth vectors, which provides a unified approach to the class of convex games and the class of assignment games with regard to the extreme core allocations.

A first reduction of a game was introduced by Davis and Maschler (1965) and since then reduced games have become a useful instrument for the analysis of several solutions for cooperative games, as is the case of Peleg's characterization of the core (1986). The reduction of an assignment game is used in Owen (1992) who shows that it may not be another assignment game, and in fact not even superadditive. This lack of superadditiveness has been the main difficulty in reaching our results.

Besides the work of Hamers *et al.* (2001), which has already been cited, Thompson (1980, 1981) analyzes efficient algorithms to compute the extreme core allocations not only for assignment games but also for transport games. There is another nice paper about the extreme core allocations of the assignment game, by Balinski and Gale (1987), which has inspired our work. They show how to check, by means of the connectedness of a graph, whether a core allocation is an extreme point. Moreover an upper bound for the number of extreme core allocations in the core of the assignment game is given and proved to be attainable. Also an attainable lower bound is provided under nondegeneracy conditions. A case of degeneracy in an assignment game is the Böhm-Bawerk's horse market where, in absence of product differentation, each buyer places the same value on any horse and, as noted in Shapley and Shubik (1971), the core consists on a line segment with two extreme allocations: one of them is optimal for the buyers and the other one is optimal for the sellers.

These two particular extreme core allocations (the sellers–optimal core allocation and the buyers–optimal core allocation) exist not only in the above particular market but in any assignment game (Shapley and Shubik, 1971).

The paper is organized as follows. Section 2 presents the formal model, the necessary notations and the expression of the core in terms of an optimal assignment. We also analyze the reduced assignment game. Roughly speaking, although not being an assignment game, it turns out to have some properties similar to those of the assignment game. The core is nonempty, in each extreme core allocation there is a player who reaches his or her marginal contribution and, on the other hand, each marginal contribution is attainable in the core.

Section 3 is devoted to the analysis of the structure of the core of the successively reduced assignment game, which turns out to be very similar to that of the core of the assignment game. This section contains the main technical results to reach the charactarization theorem of Section 4.

In Section 4, we introduce the reduced marginal worth vectors and note that, if any of them belongs to the core it must be an extreme core allocation. This is why they are good candidates to become extreme core allocations. The main result of this paper is the characterization of the set of extreme core allocations of the assignment game as the whole set of reduced marginal worth vectors (theorem 2).

2. THE FORMAL MODEL

Assignment games were introduced by Shapley and Shubik (1971) as a model for a two–sided market with transferable utility. The player set consists of the union of two finite disjoint sets $M \cup M'$, where M is the set of buyers and M' is the set of sellers. We will denote by n the cardinality of $M \cup M'$, $n = m + m'$, where m and m' are, respectively, the cardinalities of M and M'. The worth of any two–player coalition formed by a buyer $i \in M$ and a seller $j \in M'$ is $w(i,j) = a_{ij} \geq 0$. This real numbers can be arranged in a matrix and determine the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$, in the following way: $w(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S,T)\}$, being $\mathcal{M}(S,T)$ the set of matchings between S and T. A matching (or assignment) between S and T is a subset μ of $S \times T$ such that each player belongs at most to one pair in μ. It will be assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. We say a matching μ is optimal if for all $\mu' \in \mathcal{M}(M,M')$, $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$. Moreover, we say a buyer $i \in M$ is not assigned by μ if $(i,j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

The assignment model constitutes a class of cooperative game with transferable utility (TU). A TU game is a pair (N,v), where $N = \{1,2,\ldots,n\}$ is its finite player set and $v : 2^N \longrightarrow \mathbf{R}$ its characteristic function satisfying $v(\emptyset) = 0$. A payoff vector will be $x \in \mathbf{R}^n$ and, for every coalition $S \subseteq N$ we shall write $x(S) := \sum_{i\in S} x_i$ the payoff to coalition S (where $x(\emptyset) = 0$). The core of the game (N,v) consists of those payoff vectors which allocate the worth of the grand coalition in such a way that every other coalition receives at least its worth by the characteristic function: $C(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$. A game (N,v) has a nonempty core if and only if it is balanced (see Bondareva, 1963 or Shapley, 1967). From the standard classical convex analysis, we know the core is a bounded convex polyhedra. As a consequence, it has a finite number of extreme points, where we say $x \in C(v)$ is an extreme point if $y, z \in C(v)$ and $x = \frac{1}{2}y + \frac{1}{2}z$ imply $y = z$, and, moreover, the core is the convex hull of its set of extreme points. The serch of characterizations of the extreme core allocations is therefore important.

The subgame related to coalition S, $v_{|S}$, is the restriction of mapping v to the subcoalitions of S. A game is said to be superadditive when for all disjoint coalitions S and T, $v(S \cup T) \geq v(S) + v(T)$ holds. Notice that, from the definition, assignment games are always superadditive. Balanced games might not be superadditive but they always satisfy superadditive inequalities involving the grand coalition. A well known class of balanced and superadditive games is the class of convex games. A game (N, v) is convex if and only if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all pair of coalitions S and T.

The marginal contribution of player $i \in N$ in the game v, $b_i^v = v(N) - v(N \setminus \{i\})$ is an upper bound for player's i payoff in the core of the game. In general this upper bound might not be attained. However, there are balanced games with the property that all players can attain their marginal contribution in the core. This is the case of convex games and will also be the case of assignment games.

Shapley and Shubik proved that the core of the assignment game $(M \cup M', w)$ is nonempty and can be represented in terms of an optimal matching in $M \cup M'$. Let μ be one such optimal matching, then

$$C(w) = \left\{ (u,v) \in \mathbf{R}^{M \times M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M \,;\, v_j \geq 0, \text{ for all } j \in M' \\ \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu \,. \end{array} \right\} \tag{1}$$

Moreover, if for all $i \in M$, $\overline{u}_i = \max_{(u,v)\in C(w)} u_i$ and $\underline{u}_i = \min_{(u,v)\in C(w)} u_i$, while for all $j \in M'$, $\overline{v}_j = \max_{(u,v)\in C(w)} v_j$ and $\underline{v}_j = \min_{(u,v)\in C(w)} v_j$, it happens that all players on the same side of the market achieve their maximum core payoff in the same core allocation, while all players in the opposite side achieve their minimum core payoff. As a consequence, there are two special extreme core allocations: in one of them, $(\overline{u}, \underline{v})$, each buyer achieves his maximum core payoff and in the other one, $(\underline{u}, \overline{v})$, each seller does.

Demange (1982) proves that this maximum payoff of a player in the core of the assignment game is his marginal contribution, $\overline{u}_i = b_i^w = w(M \cup M') - w(M \cup M' \setminus \{i\})$ for all $i \in M$ and $\overline{v}_j = b_j^w = w(M \cup M') - w(M \cup M' \setminus \{j\})$ for all $j \in M'$. The same result was stated by Leonard (1983). The reader will also find Demange's proof in the monograph by Roth and Sotomayor (1990).

Therefore, the following one is a property assignment games have in common with convex games. Another one will be stated later.

Property 1. All marginal contributions are attained in the core of the assignment game.

The two mentioned extreme core allocations of the assignment game are not, in general, the only ones. In 1987, Balinski and Gale show how to check, in terms of the connectedness of a graph, whether a core allocation of an assignment game is in fact an extreme point. From this result follows that in each extreme core point of an assignment game there is a player who receives a zero payoff.

The reduction of a game is a well known concept in the general framework of coperative TU games. Let v be an arbitrary cooperative game with player set N and suppose some subset of players, $T \subseteq N$, is given. For a fixed vector $x \in \mathbf{R}^{N \setminus T}$, members of coalition T can reconsider their cooperative situation by means of a new game with player set T where the worth of coalitions in T is reevaluated taking into account the worth they could achieve by joining players outside T and paying them according to x. This is the reduced game of (N, v) on coalition T at x, defined by Davis and Maschler (1965):

$$v_x^T(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ \max_{\emptyset \subseteq Q \subseteq N \setminus T} \{v(S \cup Q) - x(Q)\} & \text{if } \emptyset \neq S \subset T \\ v(N) - x(N \setminus T) & \text{if } S = T \end{cases}$$

For our purposes, it will be very important to take the special case where $T = N \setminus \{i\}$, for some player $i \in N$ and $x_i = b_i^v = v(N) - v(N \setminus \{i\})$. This is what we will call the i–marginal game and denote by v^i . Marginal games were introduced in Núñez and Rafels (1998), showing their importance to analyze the extreme core allocations.

DEFINITION 1. Given a cooperative game (N,v) and a player $i \in N$ its *i–marginal game* is $(N \setminus \{i\}, v^i)$ where $v^i(\emptyset) = 0$ and for all $\emptyset \neq S \subseteq N \setminus \{i\}$,

$$v^i(S) = \max\{v(S \cup \{i\}) - b_i^v, v(S)\}\,,$$

Notice first that $v^i = v_{b_i^v}^{N\setminus\{i\}}$, since $v^i(N \setminus \{i\}) = v(N \setminus \{i\})$. Denoting by x_{-i} the restriction of x to coalition $N\setminus\{i\}$, some relationships between the core elements of (N,v) and those of its marginal games $(N \setminus \{i\}, v^i)$ are already known. From the reduced game property (RGP) of the core elements (Peleg, 1986), if $x \in C(v)$ and $x_i = b_i^v$ for some player $i \in N$, then $x_{-i} \in C(v^i)$. In fact, this relationship is also valid for extreme core elements and, together with a sort of converse property, will play an important role throughout this paper. For the sake of comprehensiveness, we state next proposition 2, the proof of which can be found in Núñez and Rafels (1998). To do this, we will denote by $x = (x_{-i}; x_i) \in \mathbf{R}^N$ the payoff vector which allocates to each player the same payoff as in $x_{-i} \in \mathbf{R}^{N\setminus\{i\}}$ and $x_i \in \mathbf{R}$ to player i .

PROPOSITION 1. *Let (N,v) be an arbitrary cooperative game,*

1. *If $x \in Ext(C(v))$ and $x_i = b_i^v$ for some $i \in N$, then , $x_{-i} \in Ext(C(v^i))$.*

2. If $x_{-i} \in Ext(C(v^i))$ *and* $v(i) \le b_i^v$ *, then* $x = (x_{-i}; b_i^v) \in Ext(C(v))$ *.*

Notice that for balanced games, conditions $v(i) \le b_i^v$, for all $i \in N$, always hold and thus

$$\bigcup_{k=1}^{n} Ext_{+k}(C(v^k)) \subseteq Ext(C(v)), \tag{2}$$

where $Ext_{+k}(C(v^k))$ denotes the set of $x \in \mathbf{R}^n$ such that $x_{-k} \in Ext(C(v^k))$ and $x_k = b_k^v$. In general, this inclusion is strict, but for some classes of games we get an equality. This is the case of assignment games.

Proposition 2. *Let* $(M \cup M', w)$ *be an assignment game, then*

$$Ext(C(w)) = \bigcup_{k=1}^{n} Ext_{+k}(C(w^k)).$$

Proof. Taking $x \in Ext(C(w))$, let us prove that there exists $k \in M \cup M'$ such that $x_k = b_k^w$. Recall first that, from Thompson (1981) and Balinski and Gale (1987), there exists $i' \in M \cup M'$ such that $x_{i'} = 0$. If this player i' is not assigned in any optimal matching of $M \cup M'$, then $b_{i'}^w = 0 = x_{i'}$. Otherwise assume, without loss of generality, that $i' \in M$ and is assigned to $j' \in M'$ by an optimal matching μ of $M \cup M'$. Then, being x a core allocation, $x(M \cup (M' \setminus \{j'\})) \ge w(M \cup (M' \setminus \{j'\}))$ and, on the other hand, as $x_{i'} = 0$,

$$x(M \cup (M' \setminus \{j'\}) = \sum_{\substack{(i,j)\in\mu \\ (i,j)\neq(i',j')}} (x_i + x_j) = \sum_{\substack{(i,j)\in\mu \\ (i,j)\neq(i',j')}} a_{ij} \le w(M \cup (M' \setminus \{j'\}),$$

as $\{(i,j) \in \mu \mid (i,j) \neq (i',j')\}$ is an assignment in $M \cup (M' \setminus \{j'\})$.

Combining both inequalities, we obtain

$$x(M \cup (M' \setminus \{j'\})) = w(M \cup (M' \setminus \{j'\}))$$

and, by efficiency, $x_{j'} = b^w_{j'}$.

We have just proved that for any $x \in Ext(C(w))$ there exists a player $k \in M \cup M'$ such that $x_k = b^w_k$. By the RGP of the extreme points of the core (part 1 of Proposition 1), $x_{-k} \in Ext(C(w^k))$ and then $x \in Ext_{+k}(C(w^k))$. ∎

Notice that the above equality also holds for convex games and thus this is another property convex games and assignment games have in common.

Property 2. In each extreme core allocation of the assignment game there is a player who is paid his marginal contribution.

Unfortunately, the reduction of an assignment game may not be another assignment game. This was already pointed out by Owen (1992) and remains true even for the particular reduced game which is the i–marginal game. Take for instance $M = \{1,2\}$, $M' = \{3,4,5\}$ and $w(i,j) = a_{ij}$ given by the matrix

	3	4	5
1	5	3	3
2	4	3	2

In the above example, the optimal matching μ is $\{(1,3),(2,4)\}$, and the core is the convex hull of three extreme points which are (3,2,2,1,0), (4,3,1,0,0) and (3,3,2,0,0).

Let us now reduce the game on coalition $M \cup M' \setminus \{4\}$ at $b_4^w = w(M \cup M') - w(M \cup (M' \setminus \{4\})) = 1$. The marginal game for player $i = 4$ is $(M \cup M' \setminus \{4\}, w^4)$ and is not superadditive ($2 = w^4(12) < w^4(1) + w^4(2) = 4$) and hence it cannot be an assignment game. Although not being an assignment game, w^4 still has a nonempty core. As $x = (3, 2, 2, 1, 0) \in C(w)$ and $x_4 = b_4^w = 1$, by part 1 of Proposition 1, $x_{-4} = (3, 2, 2, 0) \in C(w^4)$. In fact, as each player can be paid his marginal contribution in the core of the assignment game (*property 1*), all marginal games of an assignment game are balanced. In the sequel, by extending the above property to all successive reduced games we will prove balancedness holds for them all.

3. THE REDUCED ASSIGNMENT GAMES

If the reduced games had been assignment games, by applying Proposition 2 to the successive reduced games, we would obtain a natural method to express all extreme core allocations of the assignment game. In fact we will achieve the same result, in spite of them not being assignment games.

For a given ordering $\theta = (i_1, i_2, \ldots, i_n)$ on the player set $N = \{1, 2, \ldots, n\}$, we denote by v^{i_n} the i_n–marginal game of v and by $v^{i_n i_{n-1}} = (v^{i_n})^{i_{n-1}}$ the i_{n-1}–marginal game of v^{i_n}. Recursively, $v^{i_n i_{n-1} \cdots i_{k+1} i_k} = (v^{i_n i_{n-1} \cdots i_{k+1}})^{i_k}$ is the i_k–marginal game of the game $v^{i_n i_{n-1} \cdots i_{k+1}}$. The restriction of $x \in \mathbf{R}^n$ to coalition $N \setminus \{i_1, \ldots, i_k\}$ will be denoted by $x_{-i_1 i_2 \cdots i_k}$.

This section is mainly devoted to the study of the core of the successive reduced assignment game $w^{k_n k_{n-1} \cdots k_s}$, for $s \in \{2, \ldots, n\}$, in order to prove the characterization of the extreme core allocations of the assignment game in Section 4. Roughly speaking, we want to prove that all these games have a nonempty core and they also have Property 1 (page 9) and Property 2 (page 13) in common with the assignment game.

The proof of balancedness of this successive reduced game will be done by an induction argument which will be prepared along several technical lemmas.

From now on, given an optimal matching μ in $(M \cup M', w)$ and an ordering $\theta = (k_1, k_2, \ldots, k_n)$ of the player set, for any $s \in \{1, 2, \ldots, n\}$, this notation will be used: let $I_s = M \cap \{k_n, k_{n-1}, \ldots, k_s\}$, $J_s = M' \cap \{k_n, k_{n-1}, \ldots, k_s\}$, $M_s = M \setminus I_s$, $M'_s = M' \setminus J_s$ and $\mu_s = \{(i,j) \in \mu \mid i \in M_s, j \in M'_s\}$.

Notice that μ_s is the restriction to the player set $M_s \cup M'_s$ of the optimal matching μ fixed for the grand coalition $M \cup M'$.

The first one is a technical Lemma which will be used in next Lemma 2 to give a description of the core of the successive reduced assignment game $w^{k_n k_{n-1} \cdots k_s}$ in terms of a fixed optimal matching for the player set $M \cup M'$ of the original game w.

Lemma 1. *Let μ be an optimal matching for the assignment game $(M \cup M', w)$, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set and $s \in$*

$\{2, \ldots, n\}$. *If* $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ *for all* $r \in \{s, \ldots, n\}$, *and we take*

$$\begin{aligned} \forall i \in M_s\,, \qquad & \alpha_i^s := \max_{k_l \in J_s} \{0, a_{ik_l} - b_{k_l}^{w^{k_n \cdots k_{l+1}}}\} \\ \forall j \in M_s'\,, \qquad & \beta_j^s := \max_{k_l \in I_s} \{0, a_{k_l i} - b_{k_l}^{w^{k_n \cdots k_{l+1}}}\}\,, \end{aligned} \tag{3}$$

then

1. *For all* $i \in M_s$ *not assigned by* μ, $\alpha_i^s = 0$.

1'. *For all* $j \in M_s'$ *not assigned by* μ, $\beta_j^s = 0$.

2. *For all* $i \in M_s$ *not assigned by* μ_s, *but assigned to* $k_l \in J_s$ *by* μ, *it holds* $\alpha_i^s = a_{ik_l} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$.

2'. *For all* $j \in M_s'$ *not assigned by* μ_s, *but assigned to* $k_l \in I_s$ *by* μ, *it holds* $\beta_j^s = a_{k_l j} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$.

Proof. Notice first that, by hypothesis, there exists $x = (u, v) \in C(w^{k_n \cdots k_s})$ and, as $C(w^{k_n \cdots k_r}) \neq \emptyset$, for $s+1 \leq r \leq n$, by completing x with the corresponding marginal contributions, from part 2 of Proposition 1, we get a core element of the assignment game, that is

$$(x; b_{k_n}^w, b_{k_{n-1}}^{w^{k_n k_{n-1}}}, \ldots, b_{k_s}^{w^{k_n k_{n-1} \cdots k_{s+1}}}) \in C(w)\,. \tag{4}$$

By the description (1) of the core of an assignment game, if μ is an optimal matching for $M \cup M'$, then,

(i) $u_i + v_j = a_{ij}$, for all $(i,j) \in \mu_s$,

(ii) $u_i + b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} = a_{ik_l}$ if $(i,k_l) \in \mu$,

(iii) $b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} + v_j = a_{k_l j}$ if $(k_l, j) \in \mu$,

(iv) $u_i + v_j \geq a_{ij}$ for all $(i,j) \in M_s \times M'_s$, $(i,j) \notin \mu_s$,

(v) $u_i + b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} \geq a_{ik_l}$ for all $i \in M_s$ and $k_l \in J_s$,

(vi) $b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} + v_j \geq a_{k_l j}$ for all $k_l \in I_s$ and $j \in M'_s$,

(vii) $u_i \geq 0, v_j \geq 0$ for all $i \in M_s$ and $j \in M'_s$,

(viii) $u_i = 0$ for all i not matched by μ,

(ix) $v_j = 0$ for all j not matched by μ.

To prove *1*, if $i \in M_s$ is not matched by μ, then from (v) and (viii), $0 = u_i \geq a_{ik_r} - b_{k_r}^{w^{k_n k_{n-1} \cdots k_{r+1}}}$, for all $k_r \in J_s$, and then $\alpha_i^s = 0$.

To prove *2*, if $(i, k_l) \in \mu$, then, from (ii), (v) and (vii) follows that

$$u_i + b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}} = a_{ik_l}$$

$$
\begin{aligned}
u_i + b_{k_r}^{w^{k_n k_{n-1} \cdots k_{r+1}}} &\geq a_{ik_r}\,, \qquad \text{for all } k_r \in J_s \\
u_i &\geq 0\,,
\end{aligned}
$$

and consequently $\alpha_i^s = a_{ik_l} - b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$. The corresponding proofs for $j \in M'_s$ are left to the reader. ∎

The constants α_i^s and β_j^s will play an important role in the core of the reduced game $w^{k_n k_{n-1} \cdots k_s}$.

LEMMA 2. *Let μ be an optimal matching for the game $(M \cup M', w)$, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set, and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s, \ldots, n\}$, then*

$$
C(w^{k_n \cdots k_s}) = \left\{ (u,v) \in \mathbf{R}^{M_s \times M'_s} \;\middle|\; \begin{array}{l}
u_i \geq \alpha_i^s, \text{ for all } i \in M_s \\
v_j \geq \beta_j^s, \text{ for all } j \in M'_s \\
u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu_s \\
u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu_s \\
u_i = \alpha_i^s \text{ if } i \text{ not matched by } \mu_s \\
v_j = \beta_j^s \text{ if } j \text{ not matched by } \mu_s
\end{array} \right\} \qquad (5)
$$

where α_i^s , for all $i \in M_s$, and β_j^s , for all $j \in M'_s$, are defined in (3).

Proof. ($\subseteq$) We have just seen that if $(u,v) \in C(w^{k_n k_{n-1} \cdots k_s})$, the nine conditions listed in the proof of Lemma 1 hold. From conditions (v) and (vii), we get $u_i \geq \alpha_i^s$ for all $i \in M_s$ and from conditions (vi) and (vii) we get $v_j \geq \beta_i^s$ for all $j \in M'_s$. From (i) and (iv) we get $u_i + v_j = a_{ij}$ if $(i,j) \in \mu_s$ and $u_i + v_j \geq a_{ij}$ if $(i,j) \notin \mu_s$. Moreover, by Lemma 1 above, if i is not matched by μ_s , then $u_i = \alpha_i^s$ and if j is not matched by μ_s , $v_j = \beta_j^s$. Therefore, this first inclusion is proved.

($\supseteq$) Conversely, take $(u,v)\in \mathbf{R}^{M_s\times M'_s}$ satisfying all constraints defining the set in the right hand side of the equality we want to prove. By Lemma 1, as $C(w^{k_n k_{n-1}\cdots k_r})\neq\emptyset$, for all $r\in\{s,\ldots,n\}$, $\alpha_i^s=0$ if i not matched by μ and $\alpha_i^s=a_{ik_l}-b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}}$ if $(i,k_l)\in\mu$ for some $k_l\in J_s$. Similarly, $\beta_i^s=0$ if j not matched by μ and $\beta_j^s=a_{k_l j}-b_{k_l}^{w^{k_n k_{n-1}\cdots k_{l+1}}}$ if $(k_l,j)\in\mu$ for some $k_l\in I_s$. Now it is straightforward to see that $((u,v);b_{k_n}^{w},b_{k_{n-1}}^{w^{k_n}},\ldots,b_{k_s}^{w^{k_n k_{n-1}\cdots k_{s+1}}})\in C(w)$, as it fulfills all core constraints in description (1). Finally, by the reduced game property of the core elements, $(u,v)\in C(w^{k_n k_{n-1}\cdots k_s})$. ∎

It is well known that for any extreme point of the core of an assignment game there is a player with zero payoff (see Balinski and Gale (1987)). We now prove a similar property for the extreme core allocations of the reduced assignment game $w^{k_n k_{n-1}\cdots k_s}$. The result is that in every extreme core element there is a player who receives his lower bound in the representation of the core of Lemma 2.

LEMMA 3. *Let $(M\cup M',w)$ be an assignment game, $\theta=(k_1,k_2,\ldots,k_n)$ an ordering in the player set and $s\in\{2,\ldots,n\}$. If $C(w^{k_n k_{n-1}\cdots k_r})\neq\emptyset$ for all $r\in\{s,\ldots,n\}$, then for all $x=(u,v)\in Ext(C(w^{k_n k_{n-1}\cdots k_s}))$ there exists either $i\in M_s$ such that $x_i=\alpha_i^s$ or $j\in M'_s$ such that $x_j=\beta_j^s$.*

Proof. We shall consider two different cases. If there exists $i^*\in M_s$ such that i^* is not matched by μ_s, then for any $x\in C(w^{k_n k_{n-1}\cdots k_s})$, by Lemma 2, $x_{i^*}=\alpha_{i^*}^s$. Similarly, if there exists $j^*\in M'_s$ not assigned in

μ_s , then $x_{j^*} = \beta^s_{j^*}$.

Otherwise, all players in M_s are assigned to players in M'_s (and vice–versa). Assume $u_i > \alpha^s_i$ for all $i \in M_s$ and $v_j > \beta^s_j$ for all $j \in M'_s$. Then we can choose $\epsilon > 0$ such that if we define $\bar{x}$, $\bar{y} \in \mathbf{R}^{M_s \times M'_s}$,

$$\bar{x}_i = u_i + \epsilon \text{ and } \bar{y}_i = u_i - \epsilon \text{ for all } i \in M_s ,$$

$$\bar{x}_j = v_j - \epsilon \text{ and } \bar{y}_j = v_j + \epsilon \text{ for all } j \in M'_s ,$$

then $\bar{x}$ and $\bar{y}$ belong to the core of the reduced assignment game. Notice that you can choose ϵ such that $\bar{x}_i \geq \alpha^s_i$ for all $i \in M_s$, $\bar{x}_j \geq \beta^s_j$ for all $j \in M'_s$. On the other hand if $(i,j) \in M_s \times M'_s$, then $\bar{x}_i + \bar{x}_j = u_i + v_j$. As $(u,v) \in C(w)$, $\bar{x}_i + \bar{x}_j \geq a_{ij}$ if $(i,j) \notin \mu_s$ and $\bar{x}_i + \bar{x}_j = a_{ij}$ if $(i,j) \in \mu_s$. The same argument follows for vector $\bar{y}$ and then, taking the same $\epsilon > 0$ for both vectors, we obtain $\bar{x}, \bar{y} \in C(w^{k_n k_{n-1} \cdots k_s})$ and $x = \frac{1}{2}\bar{x} + \frac{1}{2}\bar{y}$, which contradicts $x = (u,v)$ being an extreme point of $C(w^{k_n k_{n-1} \cdots k_s})$. ∎

The above property allows us to prove that in each extreme allocation of the core of a successive reduced assignment game there is a player receiving his marginal contribution.

LEMMA 4. *Let* $(M \cup M', w)$ *be an assignment game,* $\theta = (k_1, k_2, \ldots, k_n)$ *an ordering in the player set and* $s \in \{2, \ldots, n\}$ *. If* $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ *for all* $r \in \{s, \ldots, n\}$ *, then, for all* $x \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ *there exists* $k \in M_s \cup M'_s$ *such that* $x_k = b_k^{w^{k_n k_{n-1} \cdots k_s}}$.

Proof. By the above Lemma, we can assume, without loss of generality, that there exists $i^* \in M_s$ such that $x_{i^*} = \alpha^s_{i^*}$. Take μ an optimal matching for $M \cup M'$ in the game w. We now consider two cases, depending on whether player i^* is matched by μ_s or not.

Case 1: Assume i^* not matched by μ_s.

As $x \in C(w^{k_n k_{n-1} \cdots k_s})$, then

$$x(M_s \cup M'_s) = w^{k_n k_{n-1} \cdots k_s}(M_s \cup M'_s)$$

and

$$x((M_s \setminus \{i^*\}) \cup M'_s) \geq w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s)\,. \tag{6}$$

On the other hand,

$$x((M_s \setminus \{i^*\}) \cup M'_s) = \sum_{(i,j)\in\mu_s} (x_i + x_j) + \sum_{\substack{i \in M_s \setminus \{i^*\} \\ i \text{ not matched by } \mu_s}} x_i + \sum_{\substack{j \in M'_s \\ j \text{ not matched by } \mu_s}} x_j\,.$$

By Lemma 1 and the core description of lemma 2,

$$x((M_s \setminus \{i^*\}) \cup M'_s) = \sum_{(i,j)\in\mu_s} a_{ij} + \tilde{a}_{p_1 k_{l_1}} - b_{k_{l_1}}^{w^{k_n k_{n-1} \cdots k_{l_1+1}}} + \cdots + \tilde{a}_{p_q k_{l_q}} - b_{k_{l_q}}^{w^{k_n k_{n-1} \cdots k_{l_q+1}}}$$

where $p_1, p_2, \ldots, p_q$ are players in $(M_s \setminus \{i^*\}) \cup M'_s$ assigned to players $k_{l_1}, k_{l_2}, \ldots, k_{l_q}$ in $I_s \cup J_s$, and we assume $l_1 > l_2 > \cdots > l_q$, and

$$\tilde{a}_{p_r k_{l_r}} = \begin{cases} a_{p_r k_{l_r}} & \text{if } p_r \in M \\ a_{k_{l_r} p_r} & \text{if } p_r \in M'\,. \end{cases}$$

Now,

$x((M_s \setminus \{i^*\}) \cup M'_s) = \sum_{(i,j)\in\mu_s} a_{ij} + \sum_{r=1}^{q} (\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1} \cdots k_{l_r+1}}}) \leq$

$w((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$

$w^{k_n \cdots k_{l_1+1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$

$w^{k_n \cdots k_{l_1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=2}^{q} b_{k_{l_r}}^{w^{k_n \cdots k_{l_r+1}}} \leq$

....

$w^{k_n \cdots k_{l_q+1}}((M_s \setminus \{i^*\}) \cup M'_s \cup \{k_{l_q}\}) - b_{k_{l_q}}^{w^{k_n \cdots k_{l_q+1}}} \leq$

$w^{k_n \cdots k_{l_q}}((M_s \setminus \{i^*\}) \cup M'_s) \leq w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s)$

where all these inequalities follow from the definition of marginal game, that is, $v^i(S) \geq v(S)$ and $v^i(S) \geq v(S \cup \{i\}) - b_i^v$, for all $S \neq \emptyset$ not containing player i. Therefore,

$$x((M_s \setminus \{i^*\}) \cup M'_s) \leq w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s).$$

We have then obtained, from (6) and the above inequality, that

$$x((M_s \setminus \{i^*\}) \cup M'_s) = w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s),$$

so, by efficiency,

$$x_i^* = w^{k_n k_{n-1} \cdots k_s}(M_s \cup M'_s) - w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i^*\}) \cup M'_s) = b_{i^*}^{w^{k_n k_{n-1} \cdots k_s}}.$$

Case 2: Player i^* is matched by μ_s to $j^* \in M'_s$.

By (3), $\alpha^s_{i^*} = \max_{k_l \in J_s}\{0, a_{i^* k_l} - b^{w^{k_n k_{n-1}\cdots k_{l+1}}}_{k_l}\}$ and thus there are two possibilities.

• If $x_{i^*} = \alpha^s_{i^*} = 0$, then on one hand, being x a core element, $x(M_s \cup M'_s) = w^{k_n k_{n-1}\cdots k_s}(M_s \cup M'_s)$ and $x(M_s \cup (M'_s \setminus \{j^*\})) \geq w^{k_n k_{n-1}\cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$.

On the other hand, by using the same reasoning and notation as in Case 1, we obtain

$x(M_s \cup (M'_s \setminus \{j^*\})) =$

$x((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) =$

$\sum_{(i,j)\in\mu_s,\,(i,j)\neq(i^*,j^*)} a_{ij} + \sum_{r=1}^{q} \left(\tilde{a}_{p_r k_{l_r}} - b^{w^{k_n k_{n-1}\cdots k_{l_r+1}}}_{k_{l_r}}\right) \leq$

$w((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\}) \cup \{k_{l_1}, k_{l_1}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b^{w^{k_n \cdots k_{l_r+1}}}_{k_{l_r}} \leq \cdots \leq$

$w^{k_n \cdots k_s}((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) \leq w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$,

where the last inequality holds by monotonicity of $w^{k_n k_{n-1}\cdots k_s}$.

We have then proved that

$$x(M_s \cup (M'_s \setminus \{j^*\})) = w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$$

and, by efficiency, $x_{j^*} = b^{w^{k_n k_{n-1}\cdots k_s}}_{j^*}$.

• If $x_{i^*} = \alpha^s_{i^*} = a_{i^* k_{l^*}} - b^{w^{k_n k_{n-1}\cdots k_{l+1}}}_{k_{l^*}}$ for some $k_{l^*} \in J_s$, there are again two possibilities. If there exists $i' \in M_s$ such that $(i', k_{l^*}) \in \mu$, then $x_{i'} =$

$\alpha^s_{i'}$ and, as i' is not assigned in M'_s, by case 1 we have $x_{i'} = b^{w^{k_n k_{n-1}\cdots k_s}}_{i'}$,

so there is a player who is paid his marginal contribution.

Otherwise, there is no $i \in M_s$ assigned to k_{l^*}. Then, as $x \in C(w^{k_n k_{n-1}\cdots k_s})$, $x(M_s \cup (M'_s \setminus \{j^*\})) \geq w^{k_n k_{n-1}\cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$. On the other hand, by an argument similar to the one above,

$$x(M_s \cup (M'_s \setminus \{j^*\})) =$$

$$x((M_s \setminus \{i^*\}) \cup (M'_s \setminus \{j^*\})) + x_{i^*} =$$

$$\sum_{(i,j)\in\mu_s,\,(i,j)\neq(i^*,j^*)} a_{ij} + \sum_{r=1}^{q} (\tilde{a}_{p_r k_{l_r}} - b^{w^{k_n k_{n-1}\cdots k_{l_r+1}}}_{k_{l_r}}) + a_{i^* k_{l^*}} - b^{w^{k_n \cdots k_{l^*+1}}}_{k_{l^*}} \leq$$

$$w(M_s \cup (M'_s \setminus \{j^*\}) \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\} \cup \{k_{l^*}\}) - \sum_{r=1}^{q} b^{w^{k_n \cdots k_{l_r+1}}}_{k_{l_r}} - b^{w^{k_n \cdots k_{l^*+1}}}_{k_{l^*}} \leq$$

$$\leq \cdots \leq$$

$$w^{k_n \cdots k_s}(M_s \cup (M'_s \setminus \{j^*\}))$$

We have then proved that

$$x(M_s \cup (M'_s \setminus \{j^*\})) = w^{k_n \cdots k_s}(M_s \cup M'_s \setminus \{j^*\}))$$

and, by efficiency, we get $x_{j^*} = b^{w^{k_n k_{n-1}\cdots k_s}}_{j^*}$. ∎

We will now focus in the structure of the core of the reduced assignment game.

Recall that the core of an assignment game has a particular structure with two special extreme points. One of them gives each seller her maximum possible payoff in the core (and so each buyer gets then his minimum possible payoff inside the core), while the other extreme gives each buyer his maximum possible payoff in the core (and so each seller gets his minimum possible payoff in the core). The next lemma states that this property is preserved in the core of the reduced assignment game.

For all $s \in \{2, \ldots, n\}$ define

$$\begin{aligned} \overline{u}_i^s &= \max\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\}, \quad \text{for all } i \in M_s\,, \\ \overline{v}_j^s &= \max\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\}, \quad \text{for all } j \in M'_s\,, \\ \underline{u}_i^s &= \min\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\}, \quad \text{for all } i \in M_s\,, \\ \underline{v}_j^s &= \min\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\}, \quad \text{for all } j \in M'_s\,. \end{aligned} \tag{7}$$

LEMMA 5. *Let $(M \cup M', w)$ be an assignment game, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s, \ldots, n\}$, then $(\overline{u}^s, \underline{v}^s)$ and $(\underline{u}^s, \overline{v}^s)$ are extreme core allocations of $w^{k_n k_{n-1} \cdots k_s}$.*

Proof. Notice first that if $(u,v), (u',v') \in C(w^{k_n k_{n-1} \cdots k_s})$ and you define for all $i \in M_s$

$$u_{*i} = \min\{u_i, u'_i\} \quad \text{and} \quad u^*_i = \max\{u_i, u'_i\}$$

and for all $j \in M'_s$

$$v_{*j} = \min\{v_j, v'_j\} \quad \text{and} \quad v^*_j = \max\{v_j, v'_j\}$$

it is easy to prove that $(u^*, v_*), (u_*, v^*) \in C(w^{k_n k_{n-1} \cdots k_s})$. We will prove it only for the point (u_*, v^*).

Let us fix an optimal matching μ of $M \cup M'$. As (u,v) and (u',v') are in the core, by Lemma 2, $u_{*i} \geq \alpha_i^s$ and $v_j^* \geq \beta_j^s$, for all $i \in M_s$, $j \in M_s'$. Moreover, either $u_{*i} + v_j^* = u_i + v_j^* \geq u_i + v_j \geq a_{ij}$ or $u_{*i} + v_j^* = u_i' + v_j^* \geq u_i' + v_j' \geq a_{ij}$.

If $(i,j) \in \mu_s$, then $u_i + v_j = a_{ij}$ and $u_i' + v_j' = a_{ij}$. Notice that if $u_i \geq u_i'$, then $v_j \leq v_j'$ and as a consequence $u_{*i} + v_j^* = u_i' + v_j' = a_{ij}$. And if $u_i \leq u_i'$, then $v_j \geq v_j'$ and $u_{*i} + v_j^* = u_i + v_j = a_{ij}$.

Lastly, if there exists $i \in M_s$ not assigned in μ_s, then, again by Lemma 2, $u_i = u_i' = \alpha_i^s$, which implies $u_{*i} = u_i^* = \alpha_i^s$.

Therefore, $(u_*, v^*) \in C(w^{k_n k_{n-1} \cdots k_s})$. Now, from the classical argument done by Shapley and Shubik (1971), it follows that $(\underline{u}^s, \overline{v}^s) \in C(w^{k_n k_{n-1} \cdots k_s})$, and in fact it is clear to see that it is an extreme core allocation. ∎

Following Roth and Sotomayor (1990), we will now see that the maximum payoff of a player in the core of the successive reduced assignment game is his marginal contribution. In fact, what we find is that the reduced assignment game also satisfies *property 2*: all marginal contributions are attained in the core of the reduced assignment game, which will be crucial for our purposes.

LEMMA 6. *Let $(M \cup M', w)$ be an assignment game, $\theta = (k_1, k_2, \ldots, k_n)$ an ordering in the player set and $s \in \{2, \ldots, n\}$. If $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s, \ldots, n\}$, then*

1. For all $i' \in M_s$, $\overline{u}^s_{i'} = b^{w^{k_n k_{n-1} \cdots k_s}}_{i'}$ *;*

2. For all $j' \in M'_s$, $\overline{v}^s_{j'} = b^{w^{k_n k_{n-1} \cdots k_s}}_{j'}$ *.*

Proof. We shall only prove the statement for all player $i' \in M_s$, as by a similar argument the reader will obtain the result for all players $j' \in M'_s$.

From Lemma 5 we know there exists $(\overline{u}^s, \underline{v}^s) \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ such that for all $i \in M_s$ and all $j \in M'_s$,

$$\overline{u}^s_i = \max\{u_i \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} \text{ and } \underline{v}^s_j = \min\{v_j \mid (u,v) \in C(w^{k_n k_{n-1} \cdots k_s})\} .$$

Take then $i' \in M_s$ and μ an optimal matching in $M \cup M'$. We now consider different cases:

Case 1: If i' not assigned in $M_s \cup M'_s$ by μ_s , then from Lemma 2 $\overline{u}^s_{i'} = \alpha^s_{i'}$ and, by the proof of case 1 of Lemma 4, $\alpha^s_{i'} = b^{w^{k_n k_{n-1} \cdots k_s}}_{i'}$.

Case 2: Otherwise i' is matched by μ_s to $j_1 \in M'_s$.

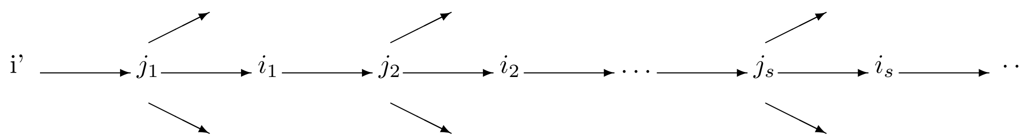
Let x be the allocation of the assignment game obtained by completing $(\overline{u}^s, \underline{v}^s)$ with the corresponding marginal contributions:

$$x = (x_k)_{k \in M \cup M'} = ((\overline{u}^s, \underline{v}^s); b^w_{k_n}, b^{w^{k_n}}_{k_{n-1}}, \cdots, b^{w^{k_n k_{n-1} \cdots k_{s+1}}}_{k_s}) .$$

By the balancedness hypothesis and repeatedly applying Proposition 1, x is a core allocation, in fact an extreme point of $C(w)$.

Construct an oriented graph with vertices $M \cup M'$ and two kind of arcs: given $(i,j) \in M \times M'$, if $(i,j) \in \mu$, then $i \longrightarrow j$ and if $x_i + x_j = a_{ij}$ but $(i,j) \notin \mu$, then $j \longrightarrow i$.

Let T be the set of $i \in M$ that can be reached from i' through an oriented path. We will assume $i' \in T$. Let S be the set of $j \in M'$ that can be reached from i' through an oriented path. Notice that, under the assumptions of case 3, neither T nor S are the empty set, as $i' \in T$ and $j_1 \in S$. Moreover, all oriented paths starting at i' pass through j_1.



We first prove that

$$\text{for all } i \notin T, \text{ and all } j \in S, \; x_i + x_j > a_{ij}. \tag{8}$$

Assume on the contrary that there exists $i \notin T$ and $j \in S$ such that $x_i + x_j = a_{ij}$ (recall $x \in C(w)$). If $(i, j) \in \mu$, then, being $j \in S$, there exists $\tilde{\imath} \in T$ such that $\tilde{\imath} \longrightarrow j$, but then $(\tilde{\imath}, j) \in \mu$ and contradicts μ being a matching. On the other hand, if $(i, j) \notin \mu$, then $j \longrightarrow i$ and as $j \in S$, we get $i \in T$ in contradiction with the hypothesis.

We now claim that "there exists an oriented path c starting at i' and ending either at $i_d \in M$ not matched by μ to a player in M'_s, or at $j_{d+1} \in S$ such that $x_{j_{d+1}} = 0$, in such a way that, in both cases, all $j \in M'$ in the path belong to M'_s". The proof of this claim will consider several cases.

If there exists some $l \in \{s, \ldots, n\}$ such that $k_l \in S$, by definition of set

S it is known to exist a path $c = (i', j_1, i_1, j_2, \ldots, j_d, i_d, k_l)$ connecting i' with k_l. Take then the path $c = (i', j_1, i_1, j_2, \ldots, j_d, i_d)$. If $j_t \in M'_s$ for all $t \in \{1, 2, \ldots, d\}$, this is the path claimed. Otherwise take $t^* = \min\{t \in \{1, 2, \ldots, d\} \mid j_t \notin M'_s\}$ and notice that $t^* > 1$ as $j_1 \in M'_s$. Take then the path $c = (i', j_1, \ldots, j_{t^*-1}, i_{t^*-1}\}$.

Assume $k_l \notin S$ for all $l \in \{s, \ldots, n\}$.

• If there exists $i_d \in T$ not matched by μ to some player in M'_s, then there is a path $c = (i', j_1, \ldots, j_d, i_d)$ with the properties claimed.

• Otherwise, all $i \in T$ are matched by μ to some player in M'_s. We will prove that there exists $k \in S$ such that $x_k = 0$.

Assume $x_k > 0$ for all $k \in S$. We then can choose $\epsilon > 0$ such that the payoff $x' \in \mathbf{R}^{M \cup M'}$ belongs to $C(w)$, where

$$\begin{aligned} x'_k &= x_k + \epsilon && k \in T \\ x'_k &= x_k - \epsilon && k \in S \\ x'_k &= x_k && k \notin S \cup T. \end{aligned}$$

You only have to take $0 < \epsilon < x_k$ for all $k \in S$ and $\epsilon < x_i + x_j - a_{ij}$ for all $j \in S$ and $i \notin T$, which is possible by claim (8). Then $x'_k \geq 0$ for all $k \in M \cup M'$ and $x'_i + x'_j \geq a_{ij}$ for $i \in M \setminus T$ and $j \in S$. Moreover, if $i \in T$ and $j \notin S$, $x'_i + x'_j = x_i + x_j + \epsilon \geq a_{ij}$. On the other hand, for $i \in T$ and $j \in S$, or $i \in M \setminus T$ and $j \in M' \setminus S$, $x'_i + x'_j = x_i + x_j$. Notice that only in the two cases $i \in T$ and $j \in S$, or $i \in M \setminus T$ and $j \in M' \setminus S$, it is possible to have $(i, j) \in \mu$ and thus all core constraints are satisfied.

- If there exists some $k_l \in T$, take $l^* = \max\{l \in \{s, \dots, n\} \mid k_l \in T\}$. If $l^* < n$, then for all $l^* \leq l \leq n$, $x'_{k_l} = x_{k_l} = b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$, and that implies, by RGP of the core, $x'_{-k_n k_{n-1} \cdots k_{l^*+1}} \in C(w^{k_n k_{n-1} \cdots k_{l^*+1}})$. But then

$$x'_{k_{l^*}} = x_{k_{l^*}} + \epsilon > x_{k_{l^*}} = b_{k_{l^*}}^{w^{k_n k_{n-1} \cdots k_{l^*+1}}}$$

which contradicts the fact that every player payoff in the core is bounded above by his marginal contribution. Notice that if $l^* = n$, then $x'_{k_n} = x_{k_n} + \epsilon = b_{k_n}^w + \epsilon > b_{k_n}^w$ and the same contradiction is reached in $C(w)$.

- Otherwise, $k_l \notin S \cup T$ for all $l \in \{s, \dots, n\}$ and then $x'_{k_l} = x_{k_l} = b_{k_l}^{w^{k_n k_{n-1} \cdots k_{l+1}}}$ for all $l \in \{s, s{+}1, \dots, n\}$. Now, as $x' \in C(w)$, by the RGP of the core, $x'_{-k_n k_{n-1} \cdots k_s} \in C(w^{k_n k_{n-1} \cdots k_s})$. But $x'_k < x_k = \underline{v}_k^s$ for all $k \in S$, in contradiction with the definition of $(\overline{u}^s, \underline{v}^s)$.

Once proved the claim, take an oriented path starting from i' and ending either at $i_d \in M$ not assigned in M'_s or at $j_{d+1} \in S$ with $x_{j_{d+1}} = 0$, and such that all $j \in S$ being in the path belong to M'_s. Let this path be $c = (i', j_1, i_1, j_2, \dots, j_d, i_d)$ or $c = (i', j_1, i_1, j_2, \dots, j_d, i_d, j_{d+1})$.

Define a matching μ' in $M_s \cup M'_s$ in the following way:

$$\mu' = \left\{ (i_t, j_t) \;\middle|\; \begin{array}{l} t \in \{1, \dots, d\} \\ i_t \in M_s \end{array} \right\} \cup \left\{ (i,j) \in M_s \times M'_s \;\middle|\; \begin{array}{l} (i,j) \in \mu_s \text{ and} \\ i \notin \{i', i_1, \dots, i_d\} \end{array} \right\} \tag{9}$$

Then, on one hand, being $(\overline{u}^s, \underline{v}^s) \in C(w^{k_n k_{n-1} \cdots k_s})$,

$$\sum_{i \in M_s,\, i \neq i'} \overline{u}_i^s + \sum_{j \in M_s'} \underline{v}_j^s \geq w^{k_n k_{n-1} \cdots k_s}((M_s \setminus \{i'\}) \cup M_s').$$

On the other hand, $\{A_1, A_2, A_3\}$ is a partition of $M_s \setminus \{i'\}$ where

$$\begin{aligned}
A_1 &= \{i_1, i_2, \ldots, i_d\} \cap M_s \\
A_2 &= \{i \in M_s \setminus \{i', i_1, \ldots, i_d\} \mid i \text{ matched by } \mu_s\} \\
A_3 &= \{i \in M_s \setminus \{i', i_1, \ldots, i_d\} \mid i \text{ not matched by } \mu_s\}
\end{aligned}$$

and $\{B_1, B_2, B_3, B_4\}$ is a partition of M_s' if $c = (i', i_1, \ldots, i_d)$ and of $M_s' \setminus \{j_{d+1}\}$ otherwise, where

$$\begin{aligned}
B_1 &= \{j_t \in \{j_1, \ldots, j_d\} \mid i_t \in M_s\} \\
B_2 &= \{j_t \in \{j_1, \ldots, j_d\} \mid i_t \notin M_s\} \\
B_3 &= \begin{cases} \{j \in M_s' \setminus \{j_1, \ldots, j_d\} \mid j \text{ matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \ldots, i_d) \\ \{j \notin \{j_1, \ldots, j_d, j_{d+1}\} \mid j \text{ matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \ldots, i_d, j_{d+1}) \end{cases} \\
B_4 &= \begin{cases} \{j \in M_s' \setminus \{j_1, \ldots, j_d\} \mid j \text{ not matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \ldots, i_d) \\ \{j \notin \{j_1, \ldots, j_d, j_{d+1}\} \mid j \text{ not matched by } \mu_s\} & \text{if } c = (i', j_1, i_1, \ldots, i_d, j_{d+1}). \end{cases}
\end{aligned}$$

By the above partitions,

$\sum_{i \in M_s,\, i \neq i'} \overline{u}_i^s + \sum_{j \in M_s'} \underline{v}_j^s =$

$\sum_{\substack{i_t \in A_1 \\ j_t \in B_1}} (\overline{u}_{i_t}^s + \underline{v}_{j_t}^s) + \sum_{\substack{i \in A_2 \\ j \in B_3 \\ (i,j) \in \mu_s}} (\overline{u}_i^s + \underline{v}_j^s) + \sum_{i \in A_3} \overline{u}_i^s + \sum_{j_t \in B_2} \underline{v}_{j_t}^s + \sum_{j \in B_4} \underline{v}_j^s + [\underline{v}_{j_{d+1}}^s]$,

where the term in brackets, [ ], is only considered when $c = (i', j_1, \ldots, i_d, j_{d+1})$ (and then $\underline{v}_{j_{d+1}}^s = 0$).

By definition of matching μ' in (9), and taking into account that players in A_2 are matched by μ_s to players in B_3 (and similarly, players in B_3 only can be matched by μ_s to players in A_2),

$$\sum_{\substack{i_t \in A_1 \\ j_t \in B_1}} (\overline{u}^s_{i_t} + \underline{v}^s_{j_t}) + \sum_{\substack{i \in A_2 \\ j \in B_3 \\ (i,j) \in \mu_s}} (\overline{u}^s_i + \underline{v}^s_j) = \sum_{(i,j) \in \mu'} a_{ij} \, .$$

A player $p \in A_3$ is either unmatched by μ, and in this case $\overline{u}^s_p = 0$, or matched by μ to a player $k_l \in J_s$, and then, by Lemma 1 and Lemma 2, $\overline{u}^s_p = a_{pk_l} - b^{w^{k_n \cdots k_{l+1}}}_{k_l}$. Similarly, if $p \in B_4$, then p is either unmatched by μ, and then $\underline{v}^s_p = 0$, or matched by μ to a player $k_l \in I_s$. In this case, again by Lemma 1 and Lemma 2, $\underline{v}^s_p = a_{k_l p} - b^{w^{k_n \cdots k_{l+1}}}_{k_l}$.

A player $j_t \in B_2$ is such that $i_t \in I_s$, that is to say, $i_t = k_l$ for some $l \in \{s, \ldots, n\}$. As j_t and i_t are connected by an arc, we get $a_{i_t j_t} = x_{i_t} + x_{j_t} = b^{w^{k_n k_{n-1} \cdots k_{l+1}}}_{k_l} + \underline{v}^s_{j_t}$ and thus $\underline{v}^s_{j_t} = a_{i_t j_t} - b^{w^{k_n k_{n-1} \cdots k_{l+1}}}_{k_l}$.

Notice finally that, if $j \in B_4$, j cannot be matched by μ to a player i_t, for $t \in \{1, \ldots, d\}$. The reason is that if $t < d$, i_t is matched by μ to j_{t+1}, while i_d is either not assigned to a player in M'_s or assigned to j_{d+1} when $c = (i', i_1, \ldots, i_d, j_{d+1})$.

Let us now denote by $p_1, p_2, \ldots, p_q$ the players in $A_3 \cup B_2 \cup B_4$ in such a way that p_r is matched by μ to k_{l_r} and $l_1 > l_2 > \cdots > l_q$. Define also

$$\tilde{a}_{p_r k_{l_r}} = \begin{cases} a_{p_r k_{l_r}} & \text{if } p_r \in M \\ a_{k_{l_r} p_r} & \text{if } p_r \in M' \, . \end{cases}$$

Now, by all the above remarks,

$\sum_{i\in M_s,\, i\neq i'} \overline{u}_i^s + \sum_{j\in M'_s} \underline{v}_j^s =$

$\sum_{(i,j)\in\mu'} a_{ij} + \sum_{r=1}^{q} \left(\tilde{a}_{p_r k_{l_r}} - b_{k_{l_r}}^{w^{k_n k_{n-1}\cdots k_{l_r+1}}} \right) \leq$

$w((M_s \setminus \{i'\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$

$w^{k_n\cdots k_{l_1+1}}((M_s \setminus \{i'\}) \cup M'_s \cup \{k_{l_1}, k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=1}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$

$w^{k_n\cdots k_{l_1}}((M_s \setminus \{i'\}) \cup M'_s \cup \{k_{l_2}, \ldots, k_{l_q}\}) - \sum_{r=2}^{q} b_{k_{l_r}}^{w^{k_n\cdots k_{l_r+1}}} \leq$

....

$w^{k_n\cdots k_{l_q+1}}((M_s \setminus \{i'\}) \cup M'_s \cup \{k_{l_q}\}) - b_{k_{l_q}}^{w^{k_n\cdots k_{l_q+1}}} \leq$

$w^{k_n\cdots k_{l_q}}((M_s \setminus \{i'\})) \cup M'_s) \leq w^{k_n\cdots k_s}((M_s \setminus \{i'\}) \cup M'_s)$

where all these inequalities follow from the definition of marginal game.

We thus have

$$\sum_{i\in M_s,\, i\neq i'} \overline{u}_i^s + \sum_{j\in M'_s} \underline{v}_j^s = w^{k_n k_{n-1}\cdots k_s}((M_s \setminus \{i'\}) \cup M'_s)$$

which, by efficiency, means that $\overline{u}_{i'}^s = b_{i'}^{w^{k_n k_{n-1}\cdots k_s}}$. ∎

Next theorem states that the successive reduced assignment games have a nonempty core and in each extreme core allocation there is a player who is paid his marginal contribution. Moreover, each player attains his marginal contribution in the core. To sum up, *property 1* and *property 2* hold for the successively reduced assignment game.

THEOREM 1. *Let $(M \cup M', w)$ be an assignment game and an arbitrary ordering $\theta = (k_1, k_2, \ldots, k_{n-1}, k_n)$ in the player set. Then, for all $s \in \{2, \ldots, n\}$:*

1. *$C(w^{k_n k_{n-1} \cdots k_s}) \neq \emptyset$*
2. *For all $x \in Ext(C(w^{k_n k_{n-1} \cdots k_s}))$ there exists $k \in M_s \cup M'_s$ such that $x_k = b_k^{w^{k_n k_{n-1} \cdots k_s}}$.*
3. *For all $i' \in M_s$, $\overline{u}^s_{i'} = b_{i'}^{w^{k_n k_{n-1} \cdots k_s}}$; and for all $j' \in M'_s$, $\overline{v}^s_{j'} = b_{j'}^{w^{k_n k_{n-1} \cdots k_s}}$,*

Proof. Since w is an assignment game, it is well known that $C(w) \neq \emptyset$ (Shapley and Shubik, 1971). Take $\theta = (k_1, \ldots, k_n)$ an arbitrary ordering in $M \cup M'$. By Property 2, given k_n there exists $x \in C(w)$, such that $x_{k_n} = b^w_{k_n}$. Now by the reduced game property of core elements, $x_{-k_n} \in C(w^{k_n})$, which proves the marginal game $C(w^{k_n})$ is balanced.

Assume iteratively that given $s+1 \in \{3, \ldots, n\}$, $C(w^{k_n k_{n-1} \cdots k_r}) \neq \emptyset$ for all $r \in \{s+1, \ldots, n\}$. By Lemma 5 and Lemma 6 there exists $x = (\overline{u}^{s+1}, \underline{v}^{s+1})$ and $y = (\underline{u}^{s+1}, \overline{v}^{s+1})$, both in $C(w^{k_n k_{n-1} \cdots k_{s+1}})$, such that if $k_s \in M_{s+1}$ then $x_{k_s} = b_{k_s}^{w^{k_n k_{n-1} \cdots k_{s+1}}}$ and if $k_s \in M'_{s+1}$ then $y_{k_s} = b_{k_s}^{w^{k_n k_{n-1} \cdots k_{s+1}}}$. In any case, there is a core element z where player k_s gets his marginal contribution in the game $w^{k_n \cdots k_{s+1}}$. Again, by the reduced game property for core allocations, $z_{-k_s} \in C(w^{k_n k_{n-1} \cdots k_{s+1} k_s})$ and thus the game $w^{k_n \cdots k_{s+1} k_s}$ is also balanced. Now statement 2 follows from Lemma 4.

Once we know the successive reduced games are all balanced, from Lemma 5 it is obtained that they all have a buyers–optimal core allocation, $(\overline{u}^s, \underline{v}^s)$, and a sellers–optimal core allocation $(\underline{u}^s, \overline{v}^s)$ (see equalities (7)). Moreover, from Lemma 6, in the core allocation of the successive reduced assignment game which is optimal for one side of the market, each agent on this side attains his marginal contribution in the corresponding game. ∎

4. THE EXTREME CORE ALLOCATIONS

For each ordering $\theta = (i_1, i_2, \ldots, i_{n-1}, i_n)$, the **reduced marginal worth vector** rm^v_θ is a vector in $\mathbf{R}^n$ where each player receives her marginal contribution to her set of predecessors, and a reduction of the game is performed in each step (Núñez and Rafels, 1998):

$$\begin{aligned}
(rm^v_\theta)_{i_n} &= v(i_1, \ldots, i_{n-1}, i_n) - v(i_1, \ldots, i_{n-1}) = b^v_{i_n}\,, \\
(rm^v_\theta)_{i_{n-1}} &= v^{i_n}(i_1, \ldots, i_{n-1}) - v^{i_n}(i_1, \ldots, i_{n-2}) = b^{v^{i_n}}_{i_{n-1}}\,, \\
&\vdots \\
(rm^v_\theta)_{i_k} &= v^{i_n i_{n-1} \cdots i_{k+1}}(i_1, \ldots, i_k) - v^{i_n i_{n-1} \cdots i_{k+1}}(i_1, \ldots, i_{k-1}) = b^{v^{i_n i_{n-1} \cdots i_{k+1}}}_{i_k}\,, \\
&\vdots \\
(rm^v_\theta)_{i_2} &= v^{i_n i_{n-1} \cdots i_3}(i_1, i_2) - v^{i_n i_{n-1} \cdots i_3}(i_1) = b^{v^{i_n i_{n-1} \cdots i_3}}_{i_2}\,, \\
(rm^v_\theta)_{i_1} &= v^{i_n i_{n-1} \cdots i_2}(i_1) = b^{v^{i_n i_{n-1} \cdots i_2}}_{i_1}\,.
\end{aligned}$$

Reduced marginal worth vectors are inspired in the marginal worth vectors where, given an ordering $\theta = (i_1, i_2, \ldots, i_n)$ on N, each player receives his marginal contribution to the set of his predecessors in the corresponding subgame, $(m_\theta^v)_{i_k} = v(i_1, \ldots, i_{k-1}, i_k) - v(i_1, \ldots, i_{k-1})$ for all $k \in \{1, 2, \ldots, n\}$.

From the definition of marginal game it follows $v^{i_n \cdots i_{k+1}}(i_1, \ldots, i_{k-1}) = v^{i_n \cdots i_k}(i_1, \ldots, i_{k-1})$ and so it is straightforward to see that reduced marginal worth vectors are always efficient, $rm_\theta^v(N) = v(N)$,

As it happens with marginal worth vectors, when a reduced marginal worth vector is a core allocation, then it is an extreme one (Núñez and Rafels, 1998). It is then easy to understand that the reduced marginal worth vectors of a given game are good candidates to become some of its extreme core allocations. The core of convex games is the convex hull of the whole set of marginal worth vectors (Shapley, 1971, and Ichiishi, 1981). Clearly, if the game is convex the reduced marginal worth vector rm_θ^v coincides with the marginal worth vector m_θ^v. However, convex games are not the only games in which the set of extreme points of the core coincide with the set of reduced marginal worth vectors. Other classes with this property can be found in Núñez and Rafels (1998). We can now state and prove that this is also the case of the assignment game. The proof relies on Theorem 1 of the previous section.

Theorem 2. *Let* $(M \cup M', w)$ *be an assignment game, then the ex-*

treme core allocations are the reduced marginal worth vectors:

$$ExtC(w) = \{rm_\theta^w\}_{\theta \in \mathcal{S}_n}$$

where $\mathcal{S}_n$ *is the set of all orderings over the player set* $M \cup M'$ *.*

Proof. Take $x \in Ext(C(w))$, by Theorem 1 there exists $k_n \in M \cup M'$ such that $x_{k_n} = b_{k_n}^w$. Now, by RGP for the extreme core points, $x_{-k_n} \in Ext(C(w^{k_n}))$. Again by Theorem 1 there exists $k_{n-1} \in (M \cup M') \setminus \{k_n\}$ such that $x_{k_{n-1}} = b_{k_{n-1}}^{w^{k_n}}$. By repeating the process, in a finite number of steps, we get an ordering $\theta = (k_1, k_2, \ldots, k_{n-1}, k_n) \in \mathcal{S}_n$ such that $x = rm_\theta^w$.

Conversely, take $x = rm_\theta^w$ for some $\theta = (k_1, k_2, \ldots, k_n)$, which means that $x_{k_{l-1}} = b_{k_{l-1}}^{w^{k_n k_{n-1} \cdots k_l}}$ for all $l \in \{2, \ldots, n\}$ and $x_{k_n} = b_{k_n}^w$. In the one–player game $w^{k_n k_{n-1} \cdots k_2}$, $x_{k_1} = b_{k_1}^{w^{k_n k_{n-1} \cdots k_2}} = w^{k_n k_{n-1} \cdots k_2}(k_1) \in Ext(C(w^{k_n k_{n-1} \cdots k_2}))$. From Theorem 1, $C(w^{k_n k_{n-1} \cdots k_3}) \neq \emptyset$, and then, by Proposition 1, we get $(x_{k_1}; b_{k_2}^{w^{k_n k_{n-1} \cdots k_3}}) \in Ext(C(w^{k_n k_{n-1} \cdots k_3}))$. By repeatedly applying Theorem 1 and Proposition 1, we get $x = rm_\theta^w \in Ext(C(w))$. ∎

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