Stable Matchings and Preferences of Couples
or
Some things couples always wanted to know about stable matchings (but were afraid to ask)*

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Abstract: It is well-known that couples that look jointly for jobs in the same centralized labor market may cause instabilities.

We demonstrate that for a natural preference domain for couples, namely the domain of responsive preferences, the existence of stable matchings can easily be established. However, a small deviation from responsiveness in one couple's preference relation that models the wish of a couple to be closer together may already cause instability. This demonstrates that the nonexistence of stable matchings in couples markets is not a singular theoretical irregularity. Our nonexistence result persists even when a weaker stability notion is used that excludes myopic blocking.

Moreover, we show that even if preferences are responsive there are problems that do not arise for singles markets. Even though for couples markets with responsive preferences the set of stable matchings is nonempty, the lattice structure that this set has for singles markets does not carry over. Furthermore we demonstrate that the new algorithm adopted by the National Resident Matching Program to fill positions for physicians in the United States may cycle, while in fact a stable matchings does exist, and be prone to strategic manipulation if the members of a couple pretend to be single.

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1 Introduction

In many countries, the proportion of women attending college has steadily been increasing during the last decades. Thus, it is not surprising that the number of couples searching jointly for a job in the same labor market has been increasing as well. So, in addition to individual job quality, couples' preferences may capture complementarities that are induced by the distance between jobs.

Roth (1984) demonstrates the possibility of instability in the presence of couples. In his example, however, the couples' preferences over pairs of positions (one position for each member of the couple) seem to be somewhat arbitrary.[3]

We show that for a natural preference domain for couples, namely the domain of responsive preferences (which reflect situations where couples search for jobs in the same metropolitan area), the existence of stable matchings can easily be established. Since the requirement of responsiveness essentially excludes complementarities in couples' preferences that are caused by distance considerations, this result – to some extend – may seem trivial. However, proceeding from this possibility result, we show that the absence of stable matchings in couples markets is not a singular theoretical irregularity: a single couple may cause a real-life market to be unstable even if their preference list is very consistently based on their individual preferences and the desire to not live too far away from each other. In other words, a small deviation from responsiveness may cause instability. Our nonexistence result persists even when we relax the requirement of stability to a weaker stability notion that excludes myopic blocking. Moreover, we show that even though for couples markets with responsive preferences the set of stable matchings is nonempty, the lattice structure that this set has for singles markets does not carry over. This for example means that the consensus that exists on each side of a singles market on which matching is worst and which one is best is lost. All the results discussed so far (Section 3) complement the theory on stability in couples markets. Moreover, they suggest that there is not much hope for a positive answer to the open question by Roth and Sotomayor (1990) whether there exist plausible classes of couples' preferences that allow for stable matchings.

In the second part of the paper (Section 4) we deal with a specific US labor market. Each year thousands of medical school graduates seek their first employment through a centralized matching process: the National Resident Matching Program (NRMP).[4] This clearinghouse was initiated in the 1950s in response to persistent failures to organize the market in a timely and orderly way by decentralized means. Around the mid 1970s voluntary and orderly participation started to drop. What happened then was that a growing number of couples in need of two positions in the same vicinity left the centralized market and started to negotiate directly with hospitals (see Checker (1973)). As a consequence, the labor market became, just as before the 1950s, prone to chaos and dissatisfaction on all sides. A hypothesis offered by Roth (1984) is that the chaotic conditions reflect the instability of the matching procedure. If couples and hospitals find it profitable to make their own arrangements outside of the matching program it must be that the matching procedure is unstable with respect to couples. This indeed turned out to be the case.

[3] Apart from possible instabilities in couples markets there are other intrinsical differences with simple matching markets, for instance (a) there may be no optimal stable matching for either side of the market (Roth (1984) and Aldershof and Carducci (1996)) or (b) different stable matchings may assign positions to different applicants and/or have different positions filled (Aldershof and Carducci (1996)).

[4] See Roth (1984), Roth and Sotomayor (1990), and Roth and Peranson (1999).

In the mid 1990s a crisis of confidence[5] in the matching procedure on the applicants side of the market finally made the NRMP Board of Directors decide to design a new algorithm. Apart from recovering confidence by favoring the students side of the market, the algorithm was also meant to deal with couples in an appropriate manner. The first match with the new algorithm was carried out in 1998. Roth and Peranson (1999) describe how the new algorithm was designed. Furthermore, using computational simulations and analyzing previous data, they show that the new algorithm is to be expected to perform well in practice.

Roth (2002) gives a more recent review of the redesign of the NRMP algorithm in the context of analyzing the "engineering aspects" of economic design. A nice overview of how the new algorithm was designed to address the problems that occur in the presence of couples is given in Roth (2002, Section 2.4.1). Roth (2002, p. 1359) reports that even though theoretically a stable matching may not exist "one result of the computational experiments conducted during the design of the algorithm is that the procedure never failed to converge to a stable matching. So there is reason to believe that the incidence of examples with no stable matchings may be rare." Furthermore, Roth (2002, p. 1359) explains that "Based on these computational experiments, the applicant proposing algorithm for the NRMP was designed so that all single candidates are admitted to the algorithm for processing before any couples are admitted. This reduces the number of times that the algorithm encounters cycles and produced the fastest convergence."

Indeed, the higher the percentage of responsive preferences for couples, the more likely the existence of a stable matching is. However, surprisingly enough, we found that even if preferences are responsive (i.e., a stable matching always exists) the new NRMP algorithm may cycle. Furthermore, since single candidates are processed first, the new NRMP algorithm may be prone to strategic manipulation by the members of a couple pretending to be single.

The relevance of the latter results in practice may need some further analysis. However, we feel that presenting these potential problems of the new NRMP algorithm contributes to understanding the complexities of the situation. We conjecture that since in this particular market students can only conduct a very small amount of interviews (fewer than 15 according to Roth (2002)), couples in order to maximize their chances of (a) receiving an offer and (b) being able to live together may focus their attention of job offers in the same region or metropolitan area, thereby segmenting the market into local markets where they can act as singles. In that case, couples preferences are likely to be responsive, which would ensure the existence of stable. This in fact may explain the observation Roth (2002, p. 1363) derives from his computations: "The computational results suggest that there may be theorems that explain why it becomes increasingly unlikely that the set of stable matchings will be either large or empty, as the market grows large." If the labor market indeed is large, it seems even more likely that couples may focus on particular regions or metropolitan areas since then these local labor markets probably are sufficiently large so that each student can conduct all interviews there. In such a situation all students can act as singles and the NRMP algorithm for singles would never fail to find the student optimal stable matching.

[5]Many students believed that the matching was not conducted in their best interest and that possibilities for strategic manipulations existed; see Roth and Peranson (1999).

2 Matching with couples: the model

For convenience and without loss of generality, we describe a model with 4 hospitals and 2 pairs of students; $H = \{h_1, h_2, h_3, h_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $C = \{c_1, c_2\} = \{(s_1, s_2), (s_3, s_4)\}$ are the sets of hospitals, students, and couples, respectively. Each hospital has exactly one position to be filled. All of our results can easily be adapted to more general situations that include other couples as well as single agents and hospitals with multiple positions.

Next, we describe preferences of hospitals, students, and couples. Each hospital $h \in H$ has a strict, transitive, and complete preference relation $\succeq_h$ over the set of students S and the prospect of having its position unfilled, denoted by $\emptyset$. Hospital h's preferences can be represented by a strict ordering of the elements in $S \cup \{\emptyset\}$; for instance, $P(h) = s_4, s_2, \emptyset, s_1, s_3$ indicates that hospital h prefers student s_4 to s_2, and considers students s_1 and s_3 to be unacceptable. In our examples and results typically each hospital prefers its position filled by some student rather than unfilled. Let $P^H = \{P(h)\}_{h \in H}$.

Similarly, each student $s \in S$ has an individual strict, transitive, and complete preference relation $\succeq_s$ over the set of hospitals and the prospect of being unemployed, denoted by u. We assume that these individual preferences are the preferences a student has if he is single. Student s's individual preferences can be represented by a strict ordering of the elements in $H \cup \{u\}$; for instance, $P(s) = h_1, h_2, h_3, h_4, u$ indicates that student s prefers h_i to h_{i+1} for $i = 1, 2, 3$ and prefers being employed to being unemployed. Let $P^S = \{P(s)\}_{s \in S}$.

Finally, each couple $c \in C$ has a strict, transitive, and complete preference relation $\succeq_c$ over all possible combination of ordered pairs of (different) hospitals and the prospect of being unemployed. Couple c's preferences can be represented by a strict ordering of the elements in $\mathcal{H} := [(H \cup \{u\}) \times (H \cup \{u\})] \backslash \{(h, h) : h \in H\}$. A generic element of $\mathcal{H}$ will be denoted by (h_p, h_q), where h_p and h_q indicate either a hospital or being unemployed. For instance, $P(c) = (h_4, h_2), (h_3, h_4), (h_4, u), etc.$ indicates that couple $c = (s_1, s_2)$ prefers s_1 and s_2 being matched to h_4 and h_2, respectively, to being matched to h_3 and h_4, respectively, and so on. Let $P^C = \{P(c)\}_{c \in C}$.

Now, the standard one-to-one matching market with single students, or *singles market* for short, is denoted by (P^H, P^S). Since singles markets and some of the classical results for singles markets are well-known, for a detailed description we refer to Roth and Sotomayor (1990) who give an excellent introduction to this model and review all results that are relevant here. For instance, the set of stable matchings is nonempty and coincides with the core. As discussed in the Introduction, some problems in real-world singles markets occurred because of the existence of couples. We define a one-to-one matching market with couples, or a *couples market* for short, by (P^H, P^C).

Before we continue, some brief remarks about our notation and its use may be helpful.

Instead of denoting a couples market by (P^H, P^C), we could add students' individual preferences and consider (P^H, P^S, P^C). Since we will not explicitly use the students' individual preferences, we suppress them in our notation.

Under certain circumstances, one can derive orderings of hospitals for both members of a couple from the couple's preferences (responsive preferences). Even though these derived preferences need not coincide with the students' individual preferences, in order to keep notation as simple as possible, we will denote the derived preferences the same way as we denote students' individual preferences.

Note that when presenting preferences in examples, we often use column notation. Furthermore, whenever we use the strict part $\succ$ of a preference relation, we assume that we compare different students, hospitals, or ordered pairs of hospitals, respectively.

Next, we introduce several possible restrictions on the couples' preferences.

First, we introduce strong unemployment aversion.[6] If couple c prefers full employment to the employment of only one partner and the employment of only one partner to the unemployment of both partners, we say that it is *strongly unemployment averse*. Formally, for all $h_p, h_q, h_r \neq u$, $(h_p, h_q) \succ_c (h_r, u) \succ_c (u, u)$ and $(h_p, h_q) \succ_c (u, h_r) \succ_c (u, u)$.

Note that a priori we do not require any relation between students' individual preferences and couples' preferences. In fact, we cannot or do not always wish to specify individual preferences when couples are concerned. However, we do study some situations in which there is a clear relationship. This is the case when the unilateral improvement of one partner's job is considered beneficial for the couple as well. Couple $c = (s_k, s_l)$ has *responsive* preferences if there exist preferences $\succeq_{s_k}$ and $\succeq_{s_l}$ such that for all $h_p, h_q, h_r \in H \cup \{u\}$, $[h_p \succ_{s_k} h_r$ implies $(h_p, h_q) \succ_c (h_r, h_q)]$ and $[h_p \succ_{s_l} h_r$ implies $(h_q, h_p) \succ_c (h_q, h_r)]$. If couple (s_k, s_l) has responsive preferences, then one can easily derive the (unique) associated individual preferences $\succeq_{s_k}$ and $\succeq_{s_l}$. Note that these associated individual preferences may in fact be identical to the students' individual preferences. However, as we will show later (see Example 3.4), associated individual preferences and students' individual preferences may not coincide.

Additive preferences are a special case of responsive preferences; couple $c = (s_k, s_l)$ has *additive* preferences if there exist functions $u_k : H \cup \{u\} \to \mathbb{R}$ and $u_l : H \cup \{u\} \to \mathbb{R}$ such that for all $h_p, h_q, h_x, h_y \in H \cup \{u\}$, $(h_p, h_q) \succ_c (h_x, h_y)$ if and only if $u_k(h_p) + u_l(h_q) > u_k(h_x) + u_l(h_y)$.

If a couple always aims to maximize the job quality for one of its members first, we say that the couple has *leader-follower* preferences. Without loss of generality we define this class of preferences for a couple $c = (s_k, s_l)$ with student s_k as the leader and student s_l as the follower: for all $h_p, h_x, h_q, h_y \in H \cup \{u\}$, $(h_p, h_x) \succ_c (h_q, h_y)$ implies $(h_p, h'_x) \succ_c (h_q, h'_y)$ for all $h'_x, h'_y \in H \cup \{u\}$.

It is easy to see that leader-follower responsive preference are additive and that not all additive preferences are of the leader-follower type. Moreover, not all responsive preferences are additive as the following example shows: $P(s_k, s_l) = (h_1, h_2), (h_2, h_1), (h_3, h_1), (h_1, h_3), (h_2, h_3), (h_3, h_2), \ldots$, where the rest of the preference list can be anything that is consistent with responsiveness. If the preference of couple (s_k, s_l) were additive we would have

$u_k(h_1) + u_l(h_2) > u_k(h_2) + u_l(h_1)$,
$u_k(h_3) + u_l(h_1) > u_k(h_1) + u_l(h_3)$, and
$u_k(h_2) + u_l(h_3) > u_k(h_3) + u_l(h_2)$,

but summing these inequalities yields a contradiction. In the next section we discuss the construction of couples' preferences out of the individual preferences in more detail.

We define an outcome μ for a couples market (P^H, P^C) to be an assignment, or *matching*, of students and positions such that each student is assigned to at most one position in H or to u (which can be assigned to multiple students), each position in H is assigned to at most one

[6] Two weaker notions of "unemployment aversion" that could be used instead of strong unemployment aversion are unemployment aversion and strict unemployment aversion: We say that couple c is *unemployment averse* if both partners being unemployed is the worst possible match for couple c. Formally, for all $(h_p, h_q) \in \mathcal{H}$, $(h_p, h_q) \succeq_c (u, u)$. If couple c always is worse off if one of its partners, or both, loose their positions, we say that it is *strictly unemployment averse*. Formally, for all $h_p, h_q \neq u$, $(h_p, h_q) \succ_c (h_p, u) \succ_c (u, u)$ and $(h_p, h_q) \succ_c (u, h_q) \succ_c (u, u)$.

student, and a student is assigned to a hospital if and only if the hospital is assigned to the student. By $\mu(S) = h_{i_1}, h_{i_2}, h_{i_3}, h_{i_4}$ we denote the positions in H or u matched to students s_1, s_2, s_3, s_4. When focusing on a particular student s_k, we will denote his match by $\mu(s_k) = h_{i_k}$. Alternatively, by $\mu(H) = s_{i_1}, s_{i_2}, s_{i_3}, s_{i_4}$ we denote the students in S or $\emptyset$ matched to hospitals h_1, h_2, h_3, h_4. When focusing on a particular hospital h_p, we will denote its match by $\mu(h_p) = s_{i_p}$. Note that the matching $\mu = (\mu(S), \mu(H))$ associated to (P^H, P^C) can be completely described either by $\mu(S)$ or by $\mu(H)$, but both notations will be useful later.[7]

Finally, we define stability for couples markets (e.g., see Roth and Sotomayor (1990)). First, for a matching to be stable, it should always be better for students to accept the position offered by the matching instead of voluntarily choosing unemployment and for hospitals it should always be better to accept the student assigned by the matching instead of leaving the position unfilled. A matching μ is *individually rational* if

(i1) for all $c = (s_k, s_l) \in C$, $(\mu(s_k), \mu(s_l)) \succeq_c (\mu(s_k), u)$, $(\mu(s_k), \mu(s_l)) \succeq_c (u, \mu(s_l))$, and $(\mu(s_k), \mu(s_l)) \succeq_c (u, u)$;

(i2) for all $h \in H$, $\mu(h) \succeq_h \emptyset$.

Second, if one partner in a couple can improve the given matching by switching to another position such that the hospital that holds the position is better off as well, then we would expect this mutually beneficial trade to be carried out, rendering the given matching instable. A similar statement holds if both students of the couple can improve. For a given matching μ, $((s_k, s_l), (h_p, h_q))$ such that $c = (s_k, s_l) \in C$ and $(h_p, h_q) \in \mathcal{H}$ is a *blocking coalition* if

(b1) $(h_p, h_q) \succ_c (\mu(s_k), \mu(s_l))$;

(b2) $[h_p \in H$ implies $s_k \succeq_{h_p} \mu(h_p)]$ and $[h_q \in H$ implies $s_l \succeq_{h_q} \mu(h_q)]$.

A matching is *stable* if it is individually rational and if there are no blocking coalitions.[8]

Roth (1984, Theorem 10) shows that stable matchings may not exist in the presence of couples. He considers the couples market (P^H, P^C) given by Table 1.

We will use the following convention for this and future examples. If $\emptyset$ is not listed for hospitals, then all students are acceptable. If the unemployment option u is not listed for students, then we have strong unemployment aversion for both couples.

By giving a blocking coalition for each of the 24 individually rational full employment matchings, Roth shows that no stable matching exists. Note that the couples' preferences are not responsive. (For couple c_1 this follows for instance from $(h_1, h_4) \succ_{c_1} (h_1, h_3)$ and $(h_2, h_3) \succ_{c_1} (h_2, h_4)$.)

In the next section, departing from Roth's example, we address one of the open questions and research directions Roth and Sotomayor (1990, p. 246, 4.) state, namely to "find reasonable assumptions about the preferences of married couples that assure the nonemptiness of the core." In other words, are there classes of "real-world preferences" for which stable matchings always exist?

[7] In our model with two couples and four hospitals we have 209 different matchings: 24 matchings with full employment, 96 matchings with one unemployed student, 72 matchings with two unemployed students, 16 matchings with three unemployed students, and 1 full unemployment matching.

[8] In order to keep notation as simple as possible, we allow some redundancy in the definition of stability with respect to (i1) and (b1).

P^H				P^C	
h_1	h_2	h_3	h_4	$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_4	s_4	s_2	s_2	h_1h_2	h_4h_2
s_2	s_3	s_3	s_4	h_4h_1	h_4h_3
s_1	s_2	s_1	s_1	h_4h_3	h_4h_1
s_3	s_1	s_4	s_3	h_4h_2	h_3h_1
				h_1h_4	h_3h_2
				h_1h_3	h_3h_4
				h_3h_4	h_2h_4
				h_3h_1	h_2h_1
				h_3h_2	h_2h_3
				h_2h_3	h_1h_2
				h_2h_4	h_1h_4
				h_2h_1	h_1h_3

Table 1: No stable matching

3 Existence of (weakly) stable matchings

First, we establish an existence result. It is based on the intuition that if there are no negative externalities from one partner's job for the other partner or for the couple, then by treating the market as if only singles participate will be sufficient to guarantee the existence of a stable matching, since this market always has an stable matching (Gale and Shapley (1962)). This would be the case if couples only apply for jobs in one city or metropolitan area so that different regional preferences or travel distance are not part of the couples' preferences anymore.

Let (P^H, P^C) be a couples market and assume that couples have responsive preferences. Then, from the couples' responsive preferences we can uniquely determine the associated individual preferences for all agents. By $(P^H, P^S(P^C))$ we denote the *associated singles market* we obtain by replacing couples in (P^H, P^C) by individual students with their associated individual preferences.

Theorem 3.1 *Let (P^H, P^C) be a couples market where couples have responsive preferences. Then, any matching that is stable for the associated singles market $(P^H, P^S(P^C))$ is also stable for (P^H, P^C). In particular, there exists a stable matching for (P^H, P^C).*

Proof. Let μ be a stable matching for $(P^H, P^S(P^C))$. Suppose that μ is not stable for (P^H, P^C). Hence, either there exists a blocking coalition or μ is not individually rational because (i1) is violated.

Assume that $((s_k, s_l), (h_p, h_q))$ is a blocking coalition. Then, (b1) $(h_p, h_q) \succ_c (\mu(s_k), \mu(s_l))$ and (b2) $[h_p \in H$ implies $s_k \succeq_{h_p} \mu(h_p)]$ and $[h_q \in H$ implies $s_l \succeq_{h_q} \mu(h_q)]$ hold.

Either $h_p \succ_{s_k} \mu(s_k)$ or $h_q \succ_{s_l} \mu(s_l)$ together with the corresponding statement of acceptability for the hospital in (b2) would contradict the stability of μ in $(P^H, P^S(P^C))$. Hence, $\mu(s_k) \succeq_{s_k} h_p$ and $\mu(s_l) \succeq_{s_l} h_q$. But then responsiveness implies $(\mu(s_k), \mu(s_l)) \succeq_c (h_p, \mu(s_l)) \succeq_c (h_p, h_q)$, which contradicts (b1).

Now assume (i1) is violated. Then there exists a couple $c = (s_k, s_l)$ such that $(\mu(s_k), u) \succ_c (\mu(s_k), \mu(s_l))$, $(u, \mu(s_l)) \succ_c (\mu(s_k), \mu(s_l))$, or $(u, u) \succ_c (\mu(s_k), \mu(s_l))$. So, $((s_k, s_l), (\mu(s_k), u))$, $((s_k, s_l), (u, \mu(s_l)))$, or $((s_k, s_l), (\mu(s_k), \mu(s_l)))$ is a blocking coalition. Using the same arguments as before we obtain a contradiction.

Hence, μ is also stable for (P^H, P^C). Finally, by Gale and Shapley (1962) a stable matching for $(P^H, P^S(P^C))$ always exists. □

The following example shows that not all stable matchings for (P^H, P^C) are stable for $(P^H, P^S(P^C))$. The intuition is that a student may not want to block by taking the position of his or her partner.

Apart from the fact that stable matchings always exist in the absence of couples, matching markets with singles have other interesting features. If preferences are strict, the set of stable matchings has a particular algebraic structure. Without specifying the details here (we refer the interested reader to the book by Roth and Sotomayor (1990)) we recall that the set of stable matchings for matching markets with singles is a distributive lattice. Example 3.2 also demonstrates that even for responsive preferences P^C the lattice structure of the set of stable matchings in $(P^H, P^S(P^C))$ need not carry over to (P^H, P^C).

Example 3.2 Consider the couples market (P^H, P^C) where the students' individual preferences P^S are given by $P(s_1) = h_4, h_1, h_2, h_3, u$, $P(s_2) = h_2, h_1, h_4, h_3, u$, $P(s_3) = h_2, h_1, h_4, h_3, u$, and $P(s_4) = h_2, h_3, h_1, h_4, u$. The preferences of the couples and the hospitals are given by Table 2. It can easily be checked that the preferences of the couples are responsive (note that $P^S = P^S(P^C)$). There are six stable matchings for the couples market (P^H, P^C): $\mu^1(S) = h_4, h_1, h_2, h_3$, $\mu^2(S) = h_4, h_1, h_3, h_2$, $\mu^3(S) = h_1, h_4, h_2, h_3$, $\mu^4(S) = h_1, h_4, h_3, h_2$, $\mu^5(S) = h_4, h_3, h_2, h_1$, and $\mu^6(S) = h_3, h_4, h_2, h_1$. However, matching $\mu^2(S)$ is the unique (hence both hospital and student optimal) stable matching for $(P^H, P^S(P^C))$. All students (hospitals) like matching $\mu^1(S)$ weakly better (worse) than all other stable matchings, but there is no agreement among the couples/students about which stable matching is worst. The latter statement shows that the set of stable matchings may not be a distributive lattice anymore if couples are present, even if they have responsive preferences. ◇

Next, we address the question whether or not one can enlarge the domain of responsive preferences while still guaranteeing the existence of stable matchings. In fact, we will start with a somewhat less ambitious task. First we relax the requirement of stability by excluding myopic behavior of blocking coalitions and ask for which reasonable preference domains weakly stable matchings always exist (see Klijn and Massó (2002) for weak stability in singles markets).

To model non-myopic behavior we assume that if the assignment of hospitals to students and students to hospitals that a blocking coalition proposes for themselves is not likely to be their final "match," then the blocking will not take place. Let μ be a matching and $((t_1, t_2), (l_1, l_2))$ be a blocking coalition. We model two cases when a blocking coalition's match most likely will not be their final match:

- the couple (t_1, t_2) that participates in the blocking coalition $((t_1, t_2), (l_1, l_2))$ can do better for themselves in another blocking coalition $(((t_1, t_2), (k_1, k_2)))$ such that other agents (one or both hospitals) that are participating in both blocking coalitions are not worse off.

P^H					P^C	
h_1	h_2	h_3	h_4		$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_4	s_4	s_2	s_2		h_4h_2	h_2h_3
s_2	s_3	s_3	s_4		h_4h_1	h_1h_2
s_1	s_2	s_1	s_1		h_1h_2	h_2h_1
s_3	s_1	s_4	s_3		h_4h_3	h_1h_3
					h_1h_4	h_2h_4
					h_2h_1	h_4h_2
					h_1h_3	h_1h_4
					h_2h_4	h_3h_2
					h_3h_2	h_4h_3
					h_2h_3	h_4h_1
					h_3h_1	h_3h_1
					h_3h_4	h_3h_4

Table 2: No lattice for responsive preferences

So, if couple (t_1, t_2) also blocks μ together with hospitals (k_1, k_2), then (t_1, t_2) prefers (k_1, k_2) to (l_1, l_2), which it would receive in the other blocking coalition, i.e., (d1) $(k_1, k_2) \succ_{(t_1,t_2)} (l_1, l_2)$.

Should any hospital be in both blocking coalitions, then it is not worse off, i.e., if for some $i, j = 1, 2$, $k_i = l_j$, then (d2) $t_i \succeq_{k_i} t_j$.

- a hospital l_p that participates in the blocking coalition $((t_1, t_2), (l_1, l_2))$ can do better for itself in another blocking coalition $(((z_1, z_2), (k_1, k_2)))$ such that other agents (the other hospital or the couple) participating in both blocking coalitions are not worse off.

  Let $l_p = k_r$, t_p be the student that is assigned to hospital l_p in blocking coalition $((t_1, t_2), (l_1, l_2))$, and z_r be the student that is assigned to hospital $k_r = l_p$ in blocking coalition $((z_1, z_2), (k_1, k_2))$.

  So, if hospital $k_r = l_p$ blocks μ together with hospital k_s ($\{k_r, k_s\} = \{k_1, k_2\}$) and couple (z_1, z_2), then it obtains a better student, i.e., (d2) $z_r \succ_{l_p} t_p$.

  Should the other hospital be in both blocking coalitions, then it should not be worse off, i.e., if for some $i, j = 1, 2$, $k_i = l_j$, then (d2) $z_i \succeq_{k_i} t_j$.

  Should the new blocking coalition be formed with the same couple, then it should not be worse off, i.e., (d1) $(k_1, k_2) \succeq_{(t_1,t_2)} (l_1, l_2)$.

We now give the formal definition. Let μ be a matching. We say that a blocking coalition $((t_1, t_2), (l_1, l_2))$ is *dominated* by another blocking coalition $((z_1, z_2), (k_1, k_2)) \neq ((t_1, t_2), (l_1, l_2))$, if

(d1) if $(z_1, z_2) = (t_1, t_2)$, then $(k_1, k_2) \succeq_{(z_1,z_2)} (l_1, l_2)$;

(d2) for all $i, j = 1, 2$, if $k_i = l_j \in H$, then $z_i \succeq_{k_i} t_j$;

(d3) $(z_1, z_2) = (t_1, t_2)$ or $k_i = l_j \in H$ for some $i, j = 1, 2$.[9]

A matching μ is *weakly stable* if it is individually rational and all blocking coalitions are dominated. Clearly, a stable matching is weakly stable. Note also that a matching with a single blocking coalition cannot be weakly stable. In some contexts it is natural to focus only on weakly stable matching with full employment (for instance when couples are strongly unemployment averse). For Roth's example (Table 1) there are three weakly stable matchings with full employment (for a proof see Appendix).

Now one might wonder whether with this weaker concept of stability we may extend the existence result in Theorem 3.1 to a larger class of preferences. For singles markets Klijn and Massó (2002) show that the set of weakly stable matchings is a superset of Zhou's (1994) bargaining set. Hence, Zhou's (1994) result that in general the bargaining set is nonempty indicates that studying weak stability might be a fruitful approach. The next theorem, however, crushes any hope for this approach.

Theorem 3.3 *Let (P^H, P^C) be a couples market where couples are strongly unemployment averse and have responsive preferences. If one couple switches two pairs of hospitals in their preference relation then no weakly stable matching with full employment may exist. In particular, no stable matching may exist after such a single switch.*

Proof. Consider a couples market with students' individual preferences $P(s_1) = h_3, h_4, h_1, h_2, u$, $P(s_2) = h_1, h_2, h_3, h_4, u$, and $P(s_3) = P(s_4) = h_2, h_1, h_3, h_4, u$. Differences in the students' individual preferences can be easily explained by "regional preferences:" even though there may exist a unanimous ranking of hospitals according to prestige or salary, students may rank certain hospitals differently because they prefer to live in a certain region, e.g., they prefer to live at the West Coast instead of at the East Coast, or vice versa.

The preferences of the strongly unemployment averse couples and the hospitals are given by Table 3. It can easily be checked that the preferences of the first couple are leader-follower responsive. The preferences of the second couple are obtained by first constructing leader-follower responsive preferences and then switching the last and second but last entries (in fact, only two hospitals for agent s_4 are switched – the switch is denoted in bold face in Table 3). This switch can be easily justified by assuming that hospital h_3 is closer than hospital h_2 to hospital h_4 which is assigned to leader s_3.

Note also that the hospitals have identical preferences over students, which can be easily justified if hospitals rank students according to final grades or other test scores. It is tedious but not difficult to check that no weakly stable matching with full employment, and therefore no stable matching, exists. This last part of the proof can be found in the Appendix. □

The example in the proof of Theorem 3.3 exhibits almost responsive preferences, except for a single switch that can easily be explained by the desire of couple (s_3, s_4) to be closer together if the leader is assigned to hospital h_4. Therefore, this example also brings us closer to answer Roth and Sotomayor's (1990) question in the negative. If we extend the domain of responsive preferences to allow for non-responsive switches that are caused by distance considerations

[9]By (d3) we ensure that we only compare conflicting blocking coalitions in the sense that there exists at least one agent that is present in both blocking coalitions. Otherwise, no domination is possible.

P^H					P^C	
h_1	h_2	h_3	h_4		$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_4	s_4	s_4	s_4		h_3h_1	h_2h_1
s_1	s_1	s_1	s_1		h_3h_2	h_2h_3
s_2	s_2	s_2	s_2		h_3h_4	h_2h_4
s_3	s_3	s_3	s_3		h_4h_1	h_1h_2
					h_4h_2	h_1h_3
					h_4h_3	h_1h_4
					h_1h_2	h_3h_2
					h_1h_3	h_3h_1
					h_1h_4	h_3h_4
					h_2h_1	$h_4\mathbf{h_3}$
					h_2h_3	h_4h_1
					h_2h_4	$h_4\mathbf{h_2}$

Table 3: Almost responsive preferences

(which is the very reason that couples may have different preferences than if they were singles), then stable matchings may not exist.

The next example reinforces this negative answer with respect to the existence of stable matchings for reasonable couples' preferences. Note that in the previous example students have different regional preferences (see explanation in the proof of Theorem 3.3), which create different individual preferences of students. The following example deals with preferences that are based on identical individual preferences of students (no differences because of regional preferences). But, in addition, we assume that if positions are too far away, the unemployment of one partner may be preferred to being separated, i.e., we drop the assumption of strong unemployment aversion. This example also illustrates how students' individual preferences may differ from the students' associated preferences as derived from the couples' preferences.[10]

Example 3.4 Consider the couples market (P^H, P^C) where the students' individual preferences P^S are given for $s \in S$ by $P(s) = h_1, h_2, h_3, h_4, u$. The hospitals' and the couples' preferences are given by Table 4. Both couples have the same preference relation. Note that as singles all students like hospital h_1 best. However, assume that hospitals h_2, h_3, and h_4 are close together, while hospital h_1 is very far away. Now, instead of being separated, the partner of a student who is matched to hospital h_1 would not accept his position because it means separation, but rather be unemployed. When ranking matchings consisting of two positions, each couple uses lexicographic preferences with respect to the quality of the position. Note that if we focus only on individually rational matchings with full employment, then the agents' preferences are responsive. In that case, a student's derived associated individual preference over hospitals

[10] Cantalà (2002) also studies the existence of stable matchings in relation to distance aspects. He shows nonexistence of stable matchings for a very restricted preference domain: he assumes that "preferences of couples satisfy the strong regional lexicographic conditions and that couples face the same geographical constraint." To be more precise, Cantalà (2002) proves his nonexistence result for matching markets containing at least six firms and three couples. By slightly changing Example 3.4 (move complete unemployment uu below h_1u in both couples' preferences in Table 4) we can extend his result to matching markets containing only four firms and two couples.

(excluding u) equals h_2, h_3, h_4, h_1. Comparing this to the student's individual preferences, we see that hospital h_1 moved from being the best position for the single student to being the worst position for the member of a couple, because working at h_1 either means separation from or unemployment of the partner.

It is easy to prove that no stable matching exists. Moreover, there is no weakly stable matching with full employment (this follows easily since any such matching is not individually rational). However, one can show for instance that the matching $\mu(S) = u, h_1, h_4, h_3$ is weakly stable. The proofs of these statements can be found in the Appendix. $\diamond$

P^H				P^C	
h_1	h_2	h_3	h_4	$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_1	s_1	s_1	s_1	h_2h_3	h_2h_3
s_3	s_3	s_3	s_3	h_2h_4	h_2h_4
s_4	s_4	s_4	s_4	h_3h_2	h_3h_2
s_2	s_2	s_2	s_2	h_3h_4	h_3h_4
				h_4h_2	h_4h_2
				h_4h_3	h_4h_3
				h_1u	h_1u
				h_2u	h_2u
				h_3u	h_3u
				h_4u	h_4u
				uh_1	uh_1
				uh_2	uh_2
				uh_3	uh_3
				uh_4	uh_4
				h_1h_2	h_1h_2
				h_1h_3	h_1h_3
				h_1h_4	h_1h_4
				h_2h_1	h_2h_1
				h_3h_1	h_3h_1
				h_4h_1	h_4h_1
				uu	uu

Table 4: Living together

4 Married couples and the new NRMP algorithm

As already mentioned in the Introduction, in the mid 1990s the American medical market was experiencing some difficulties. The proportion of students using the NRMP to obtain their positions was slowly, but consistently, decreasing over the years. The board of directors of the NRMP, responsible on behalf of a variety of medical institutions of organizing the market, asked Alvin Roth to evaluate the situation and to propose a new algorithm that could deal with one of the main source of instability: the need to link pairs of positions. The two main reasons to link positions are the presence of couples and the need of some students of being assigned to two different positions in consecutive years. A new algorithm, partially designed to take care of the instabilities produced by couples, was used by the NRMP for the first time in 1998.

The formal and complete description of the algorithm is outside the scope of this paper. Its main building block adapts (from the original one-to-one model without couples) the following dynamic process used by Roth and Vande Vate (1990). Take an individually rational matching μ and let (s, h) be a blocking pair of μ (if such a pair does not exist, μ is stable). Obtain a new matching ν from μ by satisfying the two members of the pair (s, h); namely, the matchings ν and μ coincide except that now $\nu(s) = h$, if $\mu(s) \neq u$ then $\nu(\mu(s)) = \emptyset$, and if $\mu(h) \neq \emptyset$ then $\nu(\mu(h)) = u$. Knuth (1976) showed that the process of starting at an individually rational matching and successively satisfying blocking pairs may cycle. However, Roth and Vande Vate (1990) showed that if the process cycles it is because there is at least one matching in the cycle with more than one blocking pair. In fact, they showed that for any individually rational matching μ there is a finite sequence of matchings $\mu^1, ..., \mu^K$ such that $\mu = \mu^1$, for all $1 \leq k \leq K - 1$ the matching μ^{k+1} is obtained from μ^k by satisfying a blocking pair of μ^k, and μ^K is stable.

The flowchart in the Appendix describes the parts of the new Applicant Proposing Couples Algorithm (APCA) used by the NRMP that will be relevant for understanding Examples 4.1 and 4.2 below (see Roth's web page for a complete description of the algorithm). Observe that, in contrast with the Deferred Acceptance Algorithm by Gale and Shapley (1962), this algorithm is built upon the idea of solving instabilities one at a time.

Example 4.1 shows that even for responsive preferences the algorithm might cycle without selecting a stable matching. In addition, the example also shows that this very unpleasant property of the algorithm is still aggravated because the stable matching is the outcome of the Deferred Acceptance Algorithm by Gale and Shapley (1962) and can also be obtained with probability one by (an adaptation of) the random process proposed by Roth and Vande Vate (1990).

Example 4.1 Consider the couples market (P^H, P^C) where the students' individual preferences P^S are given by $P(s_1) = P(s_2) = h_1, h_2, h_3, h_4, u$, $P(s_3) = h_2, h_1, h_3, h_4, u$, and $P(s_4) = h_3, h_4, h_2, h_1, u$. The hospitals' and the couples' preferences are given by Table 5. Note that hospitals have identical preferences over students and that the couples' preferences are leader-follower responsive.

Using Table 14 in the Appendix it can be checked easily that the unique stable matching is $\mu(H) = s_2, s_3, s_1, s_4$. Because of responsiveness it is the outcome of the Deferred Acceptance Algorithm by Gale and Shapley (1962) when students submit their individual preferences (which here coincide with the associated individual preferences).

We apply the Applicant Proposing Couples Algorithm to this couples market. Suppose that couple (s_3, s_4) is at the top of the stack (a symmetric process will occur if instead couple (s_1, s_2) is at the top of the stack). The algorithm starts at **0.0** with the empty matching $\mu^0(H) = \emptyset, \emptyset, \emptyset, \emptyset$ and cycles over the matchings $\mu^I(H) = \emptyset, s_3, s_4, \emptyset$, $\mu^{II}(H) = s_1, s_2, \emptyset, \emptyset$, $\mu^{III}(H) = s_3, \emptyset, s_4, \emptyset$, $\mu^{IV}(H) = s_2, s_1, \emptyset, \emptyset$, and finally back to $\mu^I(H)$. See the Appendix for a full, and step by step, description of the APCA applied to this couples market.

Finally, we will show that for any individually rational matching μ we can find a blocking-pair-satisfying path (i.e., à la Roth and Vande Vate (1990)) to the unique stable matching. For this we need the following notation. Given two individually rational matchings μ and μ' with full employment, we write $\mu \to \mu'$ if μ' is the consequence of 1) satisfying the blocking coalition for μ indicated in Table 14 (which leaves at most one of the other students matched), 2) unmatching the student of the other pair that is still matched (if one), and 3) assigning the students of this pair 'appropriately' to the two remaining, unmatched hospitals. For example, $\mu^2 \to \mu^{14}$ since μ^{14} follows from μ^2 by first satisfying the blocking coalition $((s_3, s_4), (h_1, h_3))$, then unmatching student s_2, and finally assigning students s_1 and s_2 to hospitals h_2 and h_4, respectively.

It is important to note that each of the three steps above is a particular instance of 'satisfying blocking coalitions'. Since μ^9 is the unique stable matching, our claim follows directly from the following observations:

$$\mu^2 \to \mu^{14} \to \mu^7 \to \mu^3 \to \mu^1 \to \mu^{13} \to \mu^9,$$
$$\mu^{11}, \mu^{12}, \mu^{17}, \mu^{18} \to \mu^7,$$
$$\mu^4, \mu^5, \mu^6, \mu^{19}, \mu^{20}, \mu^{21}, \mu^{22}, \mu^{23}, \mu^{24} \to \mu^1,$$
$$\mu^8 \to \mu^4,$$
$$\mu^{10}, \mu^{15}, \mu^{16} \to \mu^9.$$

A consequence of our claim is that any random process that begins by selecting an arbitrary matching and then proceeds to generate a sequence of matchings by satisfying blocking coalitions converges with probability one to the stable matching, provided that any blocking coalition is chosen with positive probability (that additionally only depends on the matching it blocks). ◇

P^H					P^C	
h_1	h_2	h_3	h_4		$\{s_1, s_2\}$	$\{s_3, s_4\}$
s_2	s_2	s_2	s_2		h_1h_2	h_2h_3
s_3	s_3	s_3	s_3		h_1h_3	h_2h_4
s_1	s_1	s_1	s_1		h_1h_4	h_2h_1
s_4	s_4	s_4	s_4		h_2h_1	h_1h_3
					h_2h_3	h_1h_4
					h_2h_4	h_1h_2
					h_3h_1	h_3h_4
					h_3h_2	h_3h_2
					h_3h_4	h_3h_1
					h_4h_1	h_4h_3
					h_4h_2	h_4h_2
					h_4h_3	h_4h_1

Table 5: Responsive preferences for which the new NRMP algorithm cycles

Finally, Example 4.2 illustrates the possibility that, if the APCA is used, a couple of students may obtain a better pair of positions by registering as single students rather than as a couple.

Example 4.2 Consider the couples market $\left(P^H, P^C\right)$ where hospitals h_1 and h_2 are located in one city and hospitals h_3 and h_4 are located in some other city. Assume that the two cities are very far away from each other and that the students have the same preferences over hospitals. More precisely, we assume that $P(s) = h_1, h_2, h_3, h_4, u$ for each $s \in S$ and $P(s_1, s_2) = P(s_3, s_4) = (h_1, h_2), (h_2, h_1), (h_3, h_4), (h_4, h_3), \ldots$ (the tail can be anything). In other words, the students would look for a job in another market before accepting two positions located in different cities. The hospitals' preferences over students are $P(h) = s_1, s_3, s_4, s_2, \emptyset$ for every $h \in H$.

Assume first that the four students register as couples, and couple (s_1, s_2) is at the top of the stack. Then, the APCA produces the matching $\tilde{\mu}(H) = s_1, s_2, s_3, s_4$. See the Appendix for a full, and step by step, description of the APCA applied to this couples market when the order of applicants in the stack is $(s_1, s_2), (s_3, s_4)$.

However, if s_3 and s_4 register as single students and, as a consequence, the order in the stack changes to $s_3, s_4, (s_1, s_2)$, then the algorithm produces the matching $\hat{\mu}(H) = s_3, s_4, s_1, s_2$. At this matching couple (s_3, s_4) is strictly better off than at matching $\tilde{\mu}$.[11]

See again the Appendix for a full, and step by step, description of the APCA applied to this market when the order of applicants in the stack is $s_1, s_2, (s_3, s_4)$. $\diamond$

5 Conclusion

In this paper we demonstrate that there is not much hope to find a natural real world preference domain for couples markets that would ensure the existence of stable matchings. On the positive side, we establish the existence of stable matchings if all couples have responsive preferences. We conjecture that if the labor market indeed is large, it seems very likely that couples may focus on particular regions or metropolitan areas which would induce responsiveness. In this case one could derive the associated individual preferences from the couples' preferences and apply the NRMP algorithm for singles (the student optimal Deferred Acceptance algorithm by Gale and Shapley (1962)). This algorithm would never fail to find a stable matching (particularly, it would never cycle) and furthermore, since all students can be treated as singles, the issue of manipulation by pretending to be singles instead of a couple would not arise (sequencing of course may still matter, but here at least ex ante fairness could be achieved by determining the sequence using a fair lottery).

[11] If instead the order in the stack changes to $s_4, s_3, (s_1, s_2)$, then the algorithm produces the matching $\bar{\mu}(H) = s_4, s_3, s_1, s_2$, in which couple (s_3, s_4) is also strictly better off than at matching $\tilde{\mu}$.

6 Appendix: remaining proofs

Proof of statements in Example 3.2: In Table 6 we list all 24 individually rational (full employment) matchings for the couples market with preferences given by Table 2. For the 18 matchings that are not stable we provide a blocking coalition. □

	Hospitals				Blocking coalitions?	
no.	h_1	h_2	h_3	h_4	Students	Hospitals
1	s_1	s_2	s_3	s_4	(s_3, s_4)	(h_2, h_1)
2	s_1	s_2	s_4	s_3	(s_1, s_2)	(h_4, h_2)
3	s_1	s_3	s_2	s_4	(s_1, s_2)	(h_1, h_4)
4	s_1	s_3	s_4	s_2	$--$	$--$
5	s_1	s_4	s_2	s_3	(s_1, s_2)	(h_4, h_1)
6	s_1	s_4	s_3	s_2	$--$	$--$
7	s_2	s_1	s_3	s_4	(s_3, s_4)	(h_2, h_1)
8	s_2	s_1	s_4	s_3	(s_1, s_2)	(h_4, h_2)
9	s_2	s_3	s_1	s_4	(s_3, s_4)	(h_2, h_1)
10	s_2	s_3	s_4	s_1	$--$	$--$
11	s_2	s_4	s_1	s_3	(s_1, s_2)	(h_4, h_1)
12	s_2	s_4	s_3	s_1	$--$	$--$
13	s_3	s_1	s_2	s_4	(s_1, s_2)	(h_1, h_2)
14	s_3	s_1	s_4	s_2	(s_1, s_2)	(h_1, h_2)
15	s_3	s_2	s_1	s_4	(s_1, s_2)	(h_1, h_2)
16	s_3	s_2	s_4	s_1	(s_3, s_4)	(h_2, h_3)
17	s_3	s_4	s_1	s_2	(s_1, s_2)	(h_1, h_4)
18	s_3	s_4	s_2	s_1	(s_1, s_2)	(h_4, h_1)
19	s_4	s_1	s_2	s_3	(s_1, s_2)	(h_4, h_2)
20	s_4	s_1	s_3	s_2	(s_3, s_4)	(h_2, h_1)
21	s_4	s_2	s_1	s_3	(s_1, s_2)	(h_4, h_2)
22	s_4	s_2	s_3	s_1	(s_3, s_4)	(h_2, h_1)
23	s_4	s_3	s_1	s_2	$--$	$--$
24	s_4	s_3	s_2	s_1	$--$	$--$

Table 6: Example 3.2 / Table 2, all matchings (with blocking coalitions if possible)

Proof of existence of three weakly stable matching with full employment for Roth's (1984) example: We show that for the couples market with preferences given by Table 1 there are exactly three weakly stable matchings with full employment. In Tables 7, 8, and 9 we list all 24 individually rational (full employment) matchings. Roth (1984) already showed that for all matchings at least one blocking coalition exists. (In the tables we list *all* blocking coalitions.) One can see that for the three weakly stable matchings all blocking coalitions are dominated. For the remaining matchings we indicate all undominated blocking coalitions. □

no.	Hospitals				Blocking coalitions		Undominated?
	h_1	h_2	h_3	h_4	Students	Hospitals	
1	s_1	s_2	s_3	s_4	(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
2	s_1	s_2	s_4	s_3	(s_3, s_4)	(h_4, h_2)	x
3	s_1	s_3	s_2	s_4	(s_1, s_2)	(h_1, h_4)	x
4	s_1	s_3	s_4	s_2	(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_2, h_1)	
5	s_1	s_4	s_2	s_3	(s_1, s_2)	(h_4, h_1)	x
					(s_1, s_2)	(h_4, h_3)	
					(s_1, s_2)	(h_1, h_4)	x
6	s_1	s_4	s_3	s_2	(s_3, s_4)	(h_3, h_1)	x
7	s_2	s_1	s_3	s_4	(s_1, s_2)	(h_2, h_3)	x
					(s_1, s_2)	(h_2, h_4)	
					(s_3, s_4)	(h_3, h_1)	
					(s_3, s_4)	(h_3, h_2)	
8	s_2	s_1	s_4	s_3	(s_1, s_2)	(h_4, h_1)	x
					(s_1, s_2)	(h_4, h_3)	
					(s_1, s_2)	(h_4, h_2)	
					(s_1, s_2)	(h_3, h_4)	x
					(s_1, s_2)	(h_3, h_1)	
					(s_1, s_2)	(h_3, h_2)	
					(s_1, s_2)	(h_2, h_3)	
					(s_1, s_2)	(h_2, h_4)	
					(s_3, s_4)	(h_4, h_2)	
9	s_2	s_3	s_1	s_4	(s_1, s_2)	(h_3, h_4)	
					(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_3, h_4)	
10	s_2	s_3	s_4	s_1	(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_3, h_4)	
					(s_3, s_4)	(h_2, h_4)	
					(s_3, s_4)	(h_2, h_1)	
11	s_2	s_4	s_1	s_3	(s_1, s_2)	(h_4, h_1)	x
					(s_1, s_2)	(h_4, h_3)	
					(s_1, s_2)	(h_3, h_4)	x
12	s_2	s_4	s_3	s_1	(s_3, s_4)	(h_3, h_1)	x

Table 7: Roth's (1984) example / Table 1, matchings 1-12

no.	Hospitals h_1	h_2	h_3	h_4	Blocking coalitions Students	Hospitals	Undominated?
13	s_3	s_1	s_2	s_4	(s_1, s_2)	(h_1, h_2)	
					(s_1, s_2)	(h_1, h_4)	
					(s_1, s_2)	(h_1, h_3)	
					(s_3, s_4)	(h_2, h_4)	
					(s_3, s_4)	(h_2, h_1)	
					(s_3, s_4)	(h_1, h_2)	
14	s_3	s_1	s_4	s_2	(s_1, s_2)	(h_1, h_2)	
					(s_1, s_2)	(h_1, h_4)	
					(s_1, s_2)	(h_1, h_3)	
					(s_1, s_2)	(h_3, h_4)	
					(s_1, s_2)	(h_3, h_1)	
					(s_1, s_2)	(h_3, h_2)	
					(s_1, s_2)	(h_2, h_3)	
					(s_3, s_4)	(h_3, h_1)	
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_2, h_1)	
					(s_3, s_4)	(h_2, h_3)	
					(s_3, s_4)	(h_1, h_2)	
15	s_3	s_2	s_1	s_4	(s_1, s_2)	(h_1, h_2)	
					(s_1, s_2)	(h_1, h_4)	
					(s_1, s_2)	(h_1, h_3)	
					(s_1, s_2)	(h_3, h_4)	
					(s_1, s_2)	(h_3, h_1)	
					(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_3, h_4)	
					(s_3, s_4)	(h_2, h_4)	
					(s_3, s_4)	(h_2, h_1)	
					(s_3, s_4)	(h_1, h_2)	
16	s_3	s_2	s_4	s_1	(s_1, s_2)	(h_1, h_2)	
					(s_1, s_2)	(h_4, h_1)	
					(s_1, s_2)	(h_4, h_3)	
					(s_3, s_4)	(h_3, h_1)	
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_3, h_4)	
					(s_3, s_4)	(h_2, h_4)	
					(s_3, s_4)	(h_2, h_1)	
					(s_3, s_4)	(h_2, h_3)	
					(s_3, s_4)	(h_1, h_2)	
					(s_3, s_4)	(h_1, h_4)	

Table 8: Roth's (1984) example / Table 1, matchings 13-16

no.	Hospitals h_1	h_2	h_3	h_4	Blocking coalitions Students	Hospitals	Undominated?
17	s_3	s_4	s_1	s_2	(s_1, s_2)	(h_1, h_4)	
					(s_1, s_2)	(h_1, h_3)	
					(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
18	s_3	s_4	s_2	s_1	(s_1, s_2)	(h_4, h_1)	x
19	s_4	s_1	s_2	s_3	(s_1, s_2)	(h_4, h_3)	x
					(s_1, s_2)	(h_4, h_2)	
					(s_3, s_4)	(h_4, h_2)	
20	s_4	s_1	s_3	s_2	(s_1, s_2)	(h_2, h_3)	x
21	s_4	s_2	s_1	s_3	(s_1, s_2)	(h_4, h_3)	x
					(s_1, s_2)	(h_4, h_2)	
					(s_1, s_2)	(h_3, h_4)	x
					(s_3, s_4)	(h_4, h_2)	
22	s_4	s_2	s_3	s_1	(s_1, s_2)	(h_4, h_3)	x
23	s_4	s_3	s_1	s_2	(s_3, s_4)	(h_3, h_1)	x
					(s_3, s_4)	(h_3, h_2)	
24	s_4	s_3	s_2	s_1	(s_3, s_4)	(h_2, h_4)	x

Table 9: Roth's (1984) example / Table 1, matchings 17-24

Completion of the Proof of Theorem 3.3: We still have to check that for the couples market with preferences given by Table 3 none of the 24 individually rational (full employment) matchings is weakly stable. We do this below by providing in Tables 10 and 11 at least one undominated blocking coalition for each of the matchings. □

Proof of statements in Example 3.4: To show that for the couples market defined by Table 4 no stable matching exists, let $\mathcal{H}^*$ be the seven most preferred hospital combinations depicted in Table 4, i.e.,

$$\mathcal{H}^* = \{(h_2, h_3), (h_2, h_4), (h_3, h_2), (h_3, h_4), (h_4, h_2), (h_4, h_3), (h_1, u)\}.$$

Let μ be a stable matching. Suppose that $(\mu(s_1), \mu(s_2)) \notin \mathcal{H}^*$. Then, $((s_1, s_2), (h_1, \emptyset))$ is a blocking coalition. Hence, $(\mu(s_1), \mu(s_2)) \in \mathcal{H}^*$.

Suppose that $(\mu(s_3), \mu(s_4)) \notin \mathcal{H}^*$. If $(\mu(s_1), \mu(s_2)) = (h_1, u)$, then $((s_3, s_4), (h_2, h_3))$ or $((s_3, s_4), (h_2, h_4))$ is a blocking coalition. If $(\mu(s_1), \mu(s_2)) \neq (h_1, u)$, then $((s_3, s_4), (h_1, \emptyset))$ is a blocking coalition. Hence, $(\mu(s_3), \mu(s_4)) \in \mathcal{H}^*$.

We have established that μ is one of the 12 matchings depicted in Table 12. But for each of these matchings a blocking coalition exists: a contradiction. Hence, there is no stable matching.

It remains to be proven that the (individually rational) matching $\mu(S) = u, h_1, h_4, h_3$ is weakly stable. In Table 13 we have listed all blocking coalitions for this matching, along with the blocking coalitions they are dominated by. Since each blocking coalition is dominated by some other blocking coalition it follows that μ is weakly stable. □

no.	Hospitals h_1	h_2	h_3	h_4	Blocking coalitions Students	Hospitals	Undominated?
1	s_1	s_2	s_3	s_4	(s_1, s_2)	(h_3, h_2)	x
					(s_3, s_4)	(h_3, h_2)	x
					(s_3, s_4)	(h_3, h_1)	
2	s_1	s_2	s_4	s_3	(s_1, s_2)	(h_4, h_2)	x
3	s_1	s_3	s_2	s_4	(s_1, s_2)	(h_3, h_2)	x
					(s_1, s_2)	(h_1, h_2)	
					(s_3, s_4)	(h_2, h_1)	
					(s_3, s_4)	(h_2, h_3)	
4	s_1	s_3	s_4	s_2	(s_1, s_2)	(h_4, h_2)	x
					(s_1, s_2)	(h_1, h_2)	
					(s_3, s_4)	(h_2, h_1)	
5	s_1	s_4	s_2	s_3	(s_1, s_2)	(h_3, h_4)	x
					(s_1, s_2)	(h_4, h_3)	x
					(s_3, s_4)	(h_4, h_3)	x
					(s_3, s_4)	(h_4, h_1)	
6	s_1	s_4	s_3	s_2	(s_1, s_2)	(h_3, h_4)	x
					(s_1, s_2)	(h_4, h_3)	x
					(s_1, s_2)	(h_1, h_3)	
7	s_2	s_1	s_3	s_4	(s_1, s_2)	(h_3, h_1)	x
					(s_1, s_2)	(h_1, h_3)	x
					(s_3, s_4)	(h_3, h_2)	
					(s_3, s_4)	(h_3, h_1)	
8	s_2	s_1	s_4	s_3	(s_1, s_2)	(h_4, h_1)	x
					(s_1, s_2)	(h_1, h_4)	x
9	s_2	s_3	s_1	s_4	(s_3, s_4)	(h_2, h_1)	x
					(s_3, s_4)	(h_2, h_3)	
10	s_2	s_3	s_4	s_1	(s_3, s_4)	(h_2, h_1)	x
11	s_2	s_4	s_1	s_3	(s_3, s_4)	(h_4, h_3)	x
					(s_3, s_4)	(h_4, h_1)	
12	s_2	s_4	s_3	s_1	(s_1, s_2)	(h_3, h_1)	x
13	s_3	s_1	s_2	s_4	(s_1, s_2)	(h_3, h_1)	x
					(s_1, s_2)	(h_1, h_3)	x
					(s_1, s_2)	(h_2, h_1)	
					(s_3, s_4)	(h_1, h_2)	
					(s_3, s_4)	(h_1, h_3)	

Table 10: Theorem 3.3 / Table 3, matchings 1-13 not weakly stable

no.	Hospitals h_1	h_2	h_3	h_4	Blocking coalitions Students	Hospitals	Undominated?
14	s_3	s_1	s_4	s_2	(s_1, s_2)	(h_4, h_1)	x
					(s_1, s_2)	(h_1, h_4)	x
					(s_1, s_2)	(h_2, h_1)	
					(s_3, s_4)	(h_1, h_2)	
15	s_3	s_2	s_1	s_4	(s_1, s_2)	(h_3, h_1)	x
					(s_3, s_4)	(h_1, h_2)	
					(s_3, s_4)	(h_1, h_3)	
16	s_3	s_2	s_4	s_1	(s_1, s_2)	(h_4, h_1)	x
					(s_3, s_4)	(h_1, h_2)	
17	s_3	s_4	s_1	s_2	(s_1, s_2)	(h_3, h_1)	x
18	s_3	s_4	s_2	s_1	(s_1, s_2)	(h_3, h_1)	x
					(s_1, s_2)	(h_4, h_1)	
19	s_4	s_1	s_2	s_3	(s_1, s_2)	(h_3, h_4)	x
					(s_1, s_2)	(h_4, h_3)	x
					(s_3, s_4)	(h_4, h_3)	x
20	s_4	s_1	s_3	s_2	(s_1, s_2)	(h_3, h_4)	x
					(s_1, s_2)	(h_4, h_3)	x
					(s_1, s_2)	(h_2, h_3)	
					(s_3, s_4)	(h_3, h_2)	
21	s_4	s_2	s_1	s_3	(s_3, s_4)	(h_4, h_3)	x
22	s_4	s_2	s_3	s_1	(s_1, s_2)	(h_3, h_2)	x
					(s_3, s_4)	(h_3, h_2)	x
23	s_4	s_3	s_1	s_2	(s_1, s_2)	(h_3, h_2)	x
24	s_4	s_3	s_2	s_1	(s_1, s_2)	(h_3, h_2)	x
					(s_1, s_2)	(h_4, h_2)	

Table 11: Theorem 3.3 / Table 3, matchings 14-24 not weakly stable

	Students				A blocking coalition	
no.	s_1	s_2	s_3	s_4	Students	Hospitals
1	h_1	u	h_2	h_3	(s_1, s_2)	(h_2, h_4)
2	h_1	u	h_2	h_4	(s_1, s_2)	(h_2, h_3)
3	h_1	u	h_3	h_2	(s_1, s_2)	(h_2, h_4)
4	h_1	u	h_3	h_4	(s_1, s_2)	(h_3, h_2)
5	h_1	u	h_4	h_2	(s_1, s_2)	(h_2, h_3)
6	h_1	u	h_4	h_3	(s_1, s_2)	(h_3, h_2)
7	h_2	h_3	h_1	u	(s_3, s_4)	(h_3, h_4)
8	h_2	h_4	h_1	u	(s_3, s_4)	(h_4, h_3)
9	h_3	h_2	h_1	u	(s_3, s_4)	(h_2, h_4)
10	h_3	h_4	h_1	u	(s_3, s_4)	(h_4, h_2)
11	h_4	h_2	h_1	u	(s_3, s_4)	(h_2, h_3)
12	h_4	h_3	h_1	u	(s_3, s_4)	(h_3, h_2)

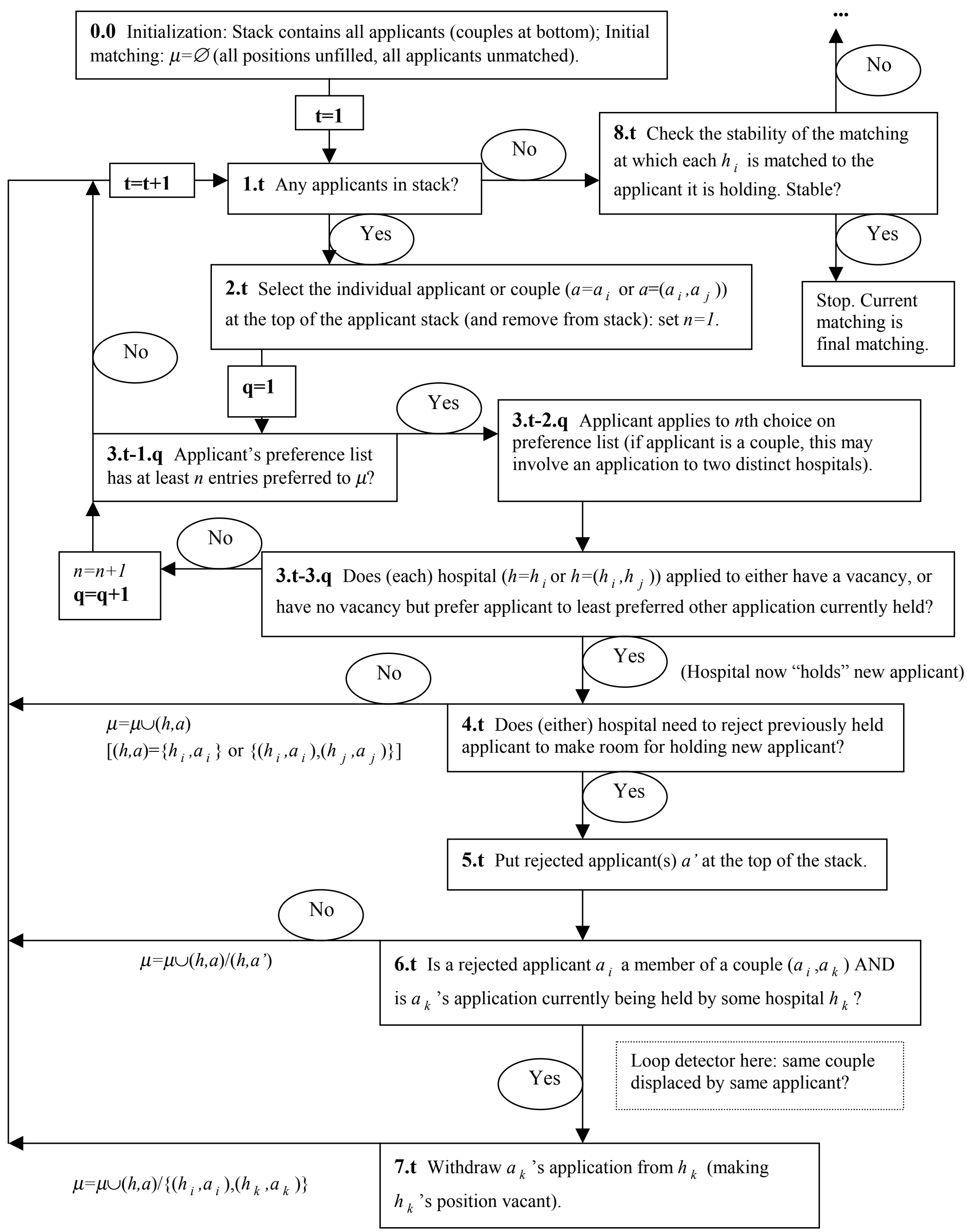
Table 12: Example 3.4 / Table 4, no stable matchings

	Blocking coalitions		
no.	Students	Hospitals	Dominated by no.
1	(s_1, s_2)	(h_3, h_2)	9
2	(s_1, s_2)	(h_4, h_2)	1
3	(s_1, s_2)	(h_1, u)	2
4	(s_1, s_2)	(h_2, u)	3
5	(s_1, s_2)	(h_3, u)	4
6	(s_1, s_2)	(h_4, u)	5
7	(s_3, s_4)	(h_2, h_3)	5
8	(s_3, s_4)	(h_3, h_2)	5
9	(s_3, s_4)	(h_4, h_2)	8

Table 13: Example 3.4 / Table 4, a weakly stable matching: $\mu(S) = u, h_1, h_4, h_3$

	Hospitals				Blocking coalitions?	
no.	h_1	h_2	h_3	h_4	Students	Hospitals
1	s_1	s_2	s_3	s_4	(s_3, s_4)	(h_1, h_4)
2	s_1	s_2	s_4	s_3	(s_3, s_4)	(h_1, h_3)
3	s_1	s_3	s_2	s_4	(s_1, s_2)	(h_1, h_2)
4	s_1	s_3	s_4	s_2	(s_1, s_2)	(h_1, h_2)
5	s_1	s_4	s_2	s_3	(s_1, s_2)	(h_1, h_2)
6	s_1	s_4	s_3	s_2	(s_1, s_2)	(h_1, h_2)
7	s_2	s_1	s_3	s_4	(s_3, s_4)	(h_2, h_4)
8	s_2	s_1	s_4	s_3	(s_3, s_4)	(h_2, h_3)
9	s_2	s_3	s_1	s_4	$--$	$--$
10	s_2	s_3	s_4	s_1	(s_1, s_2)	(h_3, h_1)
11	s_2	s_4	s_1	s_3	(s_1, s_2)	(h_2, h_1)
12	s_2	s_4	s_3	s_1	(s_1, s_2)	(h_2, h_1)
13	s_3	s_1	s_2	s_4	(s_3, s_4)	(h_2, h_4)
14	s_3	s_1	s_4	s_2	(s_1, s_2)	(h_2, h_1)
15	s_3	s_2	s_1	s_4	(s_1, s_2)	(h_3, h_1)
16	s_3	s_2	s_4	s_1	(s_1, s_2)	(h_3, h_1)
17	s_3	s_4	s_1	s_2	(s_1, s_2)	(h_2, h_1)
18	s_3	s_4	s_2	s_1	(s_1, s_2)	(h_2, h_1)
19	s_4	s_1	s_2	s_3	(s_1, s_2)	(h_1, h_2)
20	s_4	s_1	s_3	s_2	(s_1, s_2)	(h_1, h_2)
21	s_4	s_2	s_1	s_3	(s_1, s_2)	(h_1, h_2)
22	s_4	s_2	s_3	s_1	(s_1, s_2)	(h_1, h_2)
23	s_4	s_3	s_1	s_2	(s_1, s_2)	(h_1, h_2)
24	s_4	s_3	s_2	s_1	(s_1, s_2)	(h_1, h_2)

Table 14: Example 4.1 / Table 5, all matchings (with blocking coalitions if possible)



Flowchart 1: The analyzed part of the Applicant Proposing Couples Algorithm (APCA)

The APCA applied to Example 4.1:

0.0 $\mu^0(H) = \emptyset, \emptyset, \emptyset, \emptyset$.

1.1 Yes.
2.1 (s_3, s_4) is selected and set $n = 1$.
3.1-1.1 Yes, (s_3, s_4) has more than one entry preferred to $\mu^0(s_3, s_4) = (u, u)$.
3.1-2.1 (s_3, s_4) applies to (h_2, h_3).
3.1-3.1 Yes, h_2 "holds" s_3 and h_3 "holds" s_4.
4.1 No, no rejection is needed; $\mu^I(H) = \emptyset, s_3, s_4, \emptyset$.

1.2 Yes.
2.2 (s_1, s_2) is selected and set $n = 1$.
3.2-1.1 Yes, (s_1, s_2) has more than one entry preferred to $\mu^I(s_1, s_2) = (u, u)$.
3.2-2.1 (s_1, s_2) applies to (h_1, h_2).
3.2-3.1 Yes, h_1 "holds" s_1 and h_2 "holds" s_2.
4.2 Yes, h_2 rejects s_3; $\mu^{II}(H) = s_1, s_2, \emptyset, \emptyset$.

5.2 (s_3, s_4) is at the top of the stack.
6.2 s_3 is rejected and s_4 is currently being held by h_3.
7.2 s_4 is withdrawn from h_3; $\mu^{II}(H) = s_1, s_2, \emptyset, \emptyset$.

1.3 Yes.
2.3 (s_3, s_4) is selected and set $n = 1$.
3.3-1.1 Yes, (s_3, s_4) has more than one entry preferred to $\mu^{II}(s_3, s_4) = (u, u)$.
3.3-2.1 (s_3, s_4) applies to (h_2, h_3).
3.3-3.1 No, h_2 prefers s_2 to s_3. Set $n = 2$.
3.3-1.2 Yes, (s_3, s_4) has more than two entries preferred to $\mu^{II}(s_3, s_4) = (u, u)$.
3.3-2.2 (s_3, s_4) applies to (h_2, h_4).
3.3-3.2 No, h_2 prefers s_2 to s_3. Set $n = 3$.
3.3-1.3 Yes, (s_3, s_4) has more than three entries preferred to $\mu^{II}(s_3, s_4) = (u, u)$.
3.3-2.3 (s_3, s_4) applies to (h_2, h_1).
3.3-3.3 No, h_2 prefers s_2 to s_3 and h_1 prefers s_1 to s_4. Set $n = 4$.
3.3-1.4 Yes, (s_3, s_4) has more than four entries preferred to $\mu^{II}(s_3, s_4) = (u, u)$.
3.3-2.4 (s_3, s_4) applies to (h_1, h_3).
3.3-3.4 Yes, h_3 has a vacancy and h_1 prefers s_3 to s_1.
4.3 Yes, h_1 rejects s_1; $\mu^{4.3}(H) = s_3, s_2, s_4, \emptyset$.
5.3 (s_1, s_2) is at the top of the stack.
6.3 s_1 is rejected and s_2 is currently being held by h_2.
7.3 s_2 is withdrawn from h_2; $\mu^{III}(H) = s_3, \emptyset, s_4, \emptyset$.

1.4 Yes.
2.4 (s_1, s_2) is selected and set $n = 1$.
3.4-1-1 Yes, (s_1, s_2) has more than one entry preferred to $\mu^{III}(s_1, s_2) = (u, u)$.
3.4-2.1 (s_1, s_2) applies to (h_1, h_2).
3.4-3.1 No, h_1 prefers s_3 to s_1. Set $n = 2$.
3.4-1.2 Yes, (s_1, s_2) has more than two entries preferred to $\mu^{III}(s_1, s_2) = (u, u)$.
3.4-2.2 (s_1, s_2) applies to (h_1, h_3).
3.4-3.2 No, h_1 prefers s_3 to s_1. Set $n = 3$.
3.4-1.3 Yes, (s_1, s_2) has more than three entries preferred to $\mu^{III}(s_1, s_2) = (u, u)$.

3.4-2.3 (s_1, s_2) applies to (h_1, h_4).
3.4-3.3 No, h_1 prefers s_3 to s_1. Set $n = 4$.
3.4-1.4 Yes, (s_1, s_2) has more than four entries preferred to $\mu^{III}(s_1, s_2) = (u, u)$.
3.4-2.4 (s_1, s_2) applies to (h_2, h_1).
3.4-3.4 Yes, h_2 has a vacancy and h_1 prefers s_2 to s_3.
4.4 Yes, h_1 rejects s_3; $\mu^{4.4}(H) = s_2, s_1, s_4, \emptyset$.
5.4 (s_3, s_4) is at the top of the stack.
6.4 s_3 is rejected and s_4 is currently being held by h_3.
7.4 s_4 is withdrawn from h_3; $\mu^{IV}(H) = s_2, s_1, \emptyset, \emptyset$.

1.5 Yes.
2.5 (s_3, s_4) is selected and set $n = 1$.
3.5-1.1 Yes, (s_3, s_4) has more than one entry preferred to $\mu^{IV}(s_3, s_4) = (u, u)$.
3.5-2.1 (s_3, s_4) applies to (h_2, h_3).
3.5-3.1 Yes, h_3 has a vacancy and h_2 prefers s_3 to s_1.
4.5 Yes, h_2 rejects s_1; $\mu^{4.5}(H) = s_2, s_3, s_4, \emptyset$.
5.5 (s_1, s_2) is at the top of the stack.
6.5 s_1 is rejected and s_2 is currently being held by h_1.
7.5 s_1 is withdrawn from h_2; $\mu^{V}(H) = \mu^{I}(H) = \emptyset, s_3, s_4, \emptyset$.

The APCA cycles and stops at one of the unstable matchings $\mu^I, \mu^{II}, \mu^{III}, \mu^{IV}$, or μ^V.

The APCA applied to Example 4.2 with $(s_1, s_2), (s_3, s_4)$ in the stack:

0.0 $\mu^0(H) = \emptyset, \emptyset, \emptyset, \emptyset$.

1.1 Yes.
2.1 (s_1, s_2) is selected and set $n = 1$.
3.1-1.1 Yes, (s_1, s_2) has more than one entry preferred to $\mu^0(s_1, s_2) = (u, u)$.
3.1-2.1 (s_1, s_2) applies to (h_1, h_2).
3.1-3.1 Yes, h_1 "holds" s_1 and h_2 "holds" s_2.
4.1 No, no rejection is needed; $\mu^I(H) = s_1, s_2, \emptyset, \emptyset$.

1.2 Yes.
2.2 (s_3, s_4) is selected and set $n = 1$.
3.2-1.1 Yes, (s_3, s_4) has more than one entry preferred to $\mu^I(s_3, s_4) = (u, u)$.
3.2-2.1 (s_3, s_4) applies to (h_1, h_2).
3.2-3.1 No, h_1 prefers s_1 to s_3. Set $n = 2$.
3.2-1.2 Yes, (s_3, s_4) has more than two entries preferred to $\mu^I(s_3, s_4) = (u, u)$.
3.2-2.2 (s_3, s_4) applies to (h_2, h_1).
3.2-3.2 No, h_1 prefers s_1 to s_4. Set $n = 3$.
3.2-1.3 Yes, (s_3, s_4) has more than three entries preferred to $\mu^I(s_3, s_4) = (u, u)$.
3.2-2.3 (s_3, s_4) applies to (h_3, h_4).
3.2-3.3 Yes, h_3 "holds" s_3 and h_4 "holds" s_4.
4.2 No, no rejection is needed; $\mu^{II}(H) = s_1, s_2, s_3, s_4$.

1.3 No, the stack is empty.
8.3 Yes, $\tilde{\mu}(H) \equiv \mu^{II}(H) = s_1, s_2, s_3, s_4$ is stable and it is the outcome of the algorithm.

The APCA applied to Example 4.2 with $s_3, s_4, (s_1, s_2)$ in the stack:

0.0 $\mu^0(H) = \emptyset, \emptyset, \emptyset, \emptyset$.

1.1 Yes.
2.1 s_3 is selected and set $n = 1$.
3.1-1.1 Yes, s_3 has more than one entry preferred to $\mu^0(s_3) = u$.
3.1-2.1 s_3 applies to h_1.
3.1-3.1 Yes, h_1 "holds" s_3.
4.1 No, no rejection is needed; $\mu^I(H) = s_3, \emptyset, \emptyset, \emptyset$.

1.2 Yes.
2.2 s_4 is selected and set $n = 1$.
3.2-1.1 Yes, s_4 has more than one entry preferred to $\mu^I(s_4) = u$.
3.2-2.1 s_4 applies to h_1.
3.2-3.1 No, h_1 prefers s_4 to s_3. Set $n = 2$.
3.2-1.2 Yes, s_4 has more than two entries preferred to $\mu^I(s_4) = u$.
3.2-2.2 s_4 applies to h_2.
3.2-3.2 Yes, h_2 "holds" s_4.
4.2 No, no rejection is needed; $\mu^{II}(H) = s_3, s_4, \emptyset, \emptyset$.

1.3 Yes.
2.3 (s_1, s_2) is selected and set $n = 1$.
3.3-1.1 Yes, (s_1, s_2) has more than one entry preferred to $\mu^{II}(s_1, s_2) = (u, u)$.
3.3-2.1 (s_1, s_2) applies to (h_1, h_2).
3.3-3.1 No, h_2 prefers s_4 to s_2. Set $n = 2$.
3.3-1.2 Yes, (s_1, s_2) has more than two entries preferred to $\mu^{II}(s_1, s_2) = (u, u)$.
3.3-2.2 (s_1, s_2) applies to (h_2, h_1).
3.3-3.2 No, h_1 prefers s_3 to s_2. Set $n = 3$.
3.3-1.3 Yes, (s_1, s_2) has more than three entries preferred to $\mu^{II}(s_1, s_2) = (u, u)$.
3.3-2.3 (s_1, s_2) applies to (h_3, h_4).
3.3-3.3 Yes, h_3 "holds" s_1 and h_4 "holds" s_2.
4.3 No, no rejection is needed; $\mu^{III}(H) = s_3, s_4, s_1, s_2$.
1.4 No, the stack is empty.
8.4 Yes, $\hat{\mu}(H) \equiv \mu^{III}(H) = s_3, s_4, s_1, s_2$ is stable and it is the outcome of the algorithm.

References

Aldershof, B. and Carducci, O.M. (1996) "Stable Matchings with Couples," *Discrete Applied Mathematics*, **68**, 203-207.

Cantalà, D. (2002) "Matching Markets: the Particular Case of Couples," Mimeo, Department of Economics, ITAM, Mexico

Checker, A. (1973) "The National Intern and Resident Matching Program, 1966-1972," *Journal of Medical Education*, **48**, 107-109.

Gale, D. and Shapley, L.S. (1962) "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, **69**, 9-15.

Klijn, F. and Massó, J. (2002) "Weak Stability and a Bargaining Set for the Marriage Model," forthcoming in *Games and Economic Behavior*

Knuth, D.E. (1976) Marriages Stables. Montreal: Les Presses de l'Universite Montreal.

Roth, A.E. (1984) "The Evolution of the Labor Market for Medical Interns and Residents: a Case Study in Game Theory," *Journal of Political Economy*, **92**, 991-1016.

Roth, A.E. (2002) "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics," *Econometrica*, **70**, 1341-1378.

Roth, A.E. and Peranson, E. (1999) "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design," *American Economic Review*, **89**, 748-780.

Roth, A.E. and Sotomayor, M. (1990) Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Econometric Society Monograph Series. New York: Cambridge University Press.

Roth, A.E. and Vande Vate, J.H. (1990) "Random Paths to Stability in Two-Sided Matching," *Econometrica*, **58**, 1475-1480.

Zhou, L. (1994) "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation," *Games and Economic Behavior*, **6**, 512-526.