Analogy-Based Expectation Equilibrium*

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Abstract

In complicated interactions, agents use simplified representations to learn how their environment may react. I assume that agents bundle nodes at which other agents must move into analogy classes, and agents only try to learn the average behavior in every class. Specifically, I propose a new solution concept for multi-stage games with perfect information: at every node players choose best-responses to their analogy-based expectations, and expectations correctly represent the average behavior in every class. The solution concept is contrasted with other solution concepts, and it is applied to a variety of games. It is shown that a player may benefit from having a coarse analogy partitioning. And by contrast with the standard approach, (1) initial cooperation followed by an end opportunistic behavior may emerge in the finitely repeated prisoner's dilemma, (2) the responder in a take-it-or-leave-it offer game may get a payoff much above his reservation value.

Key words: Game theory, bounded rationality, expectation, reasoning by analogy.

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1 Introduction

Received game theory assumes that players are perfectly rational both in their ability to form *correct* expectations about other players' behavior and in their ability to select *best-responses* to their expectations. The game of chess is a striking example in which the standard approach is inappropriate. In chess, it is clearly impossible to know (learn) what the opponent might do in any event (i.e. at every board position).

In complicated interactions, players or agents use simplified representations to learn how their environment may react. Specifically, I assume that players *bundle* nodes at which other players must move into *analogy classes*. And every player only tries to learn the *average* behavior in each analogy class that he considers (as opposed to trying to learn the behavior for every single contingency in which other players must move). In equilibrium I require that (1) players play best-responses to their expectations, and (2) expectations are correct for every class (that is, the expectation attached to every analogy class correctly represents the average behavior in the class).

To illustrate the implication of the approach, consider a variant of the finitely repeated prisoner's dilemma in which there are many periods, there is no discounting and the exact values of the stage game prisoner's dilemma payoffs are independently drawn from period to period according to some pre-specified distribution (with finite support).[1]

The standard approach assumes that players can make correct expectations about their opponent's behavior after any conceivable history of play. And it predicts that players must behave opportunistically in all periods.[2] Suppose, by contrast, that players have the simplified representation that the key determinant to their opponent's behavior is whether or not some opportunistic behavior was previously observed in the interaction. Based on that representation, players form expectations about their opponent's behavior only according to whether or not some opportunistic behavior appears in past play.

Playing cooperatively most of the time except if some opportunistic behavior previously occurred or toward the end of the game (if the immediate gain from switching to an opportunistic behavior is sufficiently high) is part of an equilibrium when players have such a simplified representation in mind.

To see this, suppose that players do behave as explained above. The correctness of expectations implies that each player should *expect* the other player (1) to play opportunistically

[1]Except for subjects thinking in terms of backward induction (who are rare according to Johnson et al. 2002), such a game seems sufficiently complicated for the current approach to have some bite.

[2]In the last period, opportunistic behavior is a dominant strategy. Backwards induction reveals that no other behavior can emerge earlier.

whenever some opportunistic behavior previously occurred and (2) to play cooperatively (on average) with a large probability otherwise (the number of repetitions is large). Given such expectations, playing cooperatively in all but a few periods toward the end is optimal, as long as no opportunistic behavior appears in past play.[3] And when some opportunistic behavior appears, it is optimal to play opportunistically. This line of argument shows that cooperating most of the time is an equilibrium in our setting.

The aim of this paper is twofold. The first objective is to propose a solution concept to describe the interaction of players forming their expectations by analogy. This will be called the analogy-based expectation equilibrium. The second objective is to analyze the properties of analogy-based expectation equilibria in various strategic interaction contexts, and show how new phenomena may arise.

The games we consider are multi-stage games with almost perfect information and perfect recall. That is, simultaneous moves and moves by Nature are allowed. But, in any stage, all previous moves are assumed to be known to every player. The partitioning into analogy classes used by the players is given exogenously, and it stands for a reduced form of how players simplify the expectation problem to make learning manageable.[4] The analogy partitioning is viewed as part of the description of the strategic environment. An analogy class α_i of player i is a set of pairs (j, h) such that player j, $j \neq i$, must move at node h. We require that if two elements (j, h) and (j', h') belong to the same analogy class, the action spaces of player j at node h and of player j' at node h' are identically labelled.

Player i's *analogy-based expectation* β_i is player i's expectation about the average behavior of other players in every analogy class α_i considered by player i - we will denote by $\beta_i(\alpha_i)$ the expectation in the analogy class α_i. An *analogy-based expectation equilibrium* is a pair (σ, β) where σ is a strategy profile and β is an analogy-based expectation profile such that two conditions are satisfied. First, for each player i and for each node at which player i must move, player i's strategy σ_i is a *best-response* to his analogy-based expectation β_i.[5] Second, for each player i and analogy class α_i, player i's expectation $\beta_i(\alpha_i)$ *correctly* represents the average behavior in class α_i as induced by the strategy profile σ (where the behavior of player

[3] This is so because players perceive that by playing opportunistically they will trigger a non-cooperative phase whereas by playing cooperatively they expect the other player to continue playing cooperatively with a large probability.

[4] One might think of the partitioning as resulting from the past experiences of the players and also from the way the strategic interaction is framed to the players thus triggering some connections with past experiences (the so called framing effect, see Tversky-Khaneman 1981).

[5] More precisely, player i's strategy σ_i is a best-response (at every node where player i must move) to the behavioral strategy that assigns player j to play according to the expectation $\beta_i(\alpha_i)$ at node h, for every (j, h) in the analogy class α_i and for every analogy class α_i.

j in node h , $(j,h) \in \alpha_i$, is weighted by the frequency with which (j,h) is visited according to σ - relative to other elements in α_i).[6]

The setup captures the following aspects of analogy-based reasoning. First, as the common sense of analogy reasoning suggests, several problems (here expectations) are dealt with together by every player.[7] Second, the correctness of expectations implies that, in any given class, contingencies which are visited more often contribute more to the expectation than contingencies which are visited less often. Accordingly, the behaviors in frequently visited contingencies contaminate the expectation used in all contingencies of the analogy class no matter how often they are visited. The extrapolation (here of the expectation) from more visited to less visited contingencies is - we believe - a key feature of the analogy idea.

Interestingly, the analogy-based expectation equilibrium *cannot* be viewed as a standard equilibrium (say a sequential equilibrium) of another game even by varying the information structure (while keeping the payoff and move structures)[8]. So it is an entirely new solution concept whose properties need to be investigated. In the special case where all players use the finest partitioning as their analogy devices, the strategy profile of an analogy-based expectation equilibrium coincides with a Subgame Perfect Nash equilibrium. But, otherwise, the play of an analogy-based expectation equilibrium will in general differ from that of a Subgame Perfect Nash equilibrium or even from that of a Nash equilibrium.[9]

In the second part of the paper, we apply the concept to a variety of games.[10] We first observe through an example that sometimes a player may *benefit* from having a coarse analogy partitioning as compared with the finest partitioning.[11] The example also serves to illustrate

[6]We think of the consistency requirement as resulting from a learning process in which players would eventually manage to have correct analogy-based expectations (and not as resulting from introspection or calculations on the part of the players). And if no node h such that (j,h) belongs to α_i is ever visited according to σ, (strong) consistency is defined with respect to a small perturbation of σ. (This is in spirit of the definition of sequential equilibrium.)

[7]Every player pools together several contingencies in order to form an aggregate expectation about the average behavior (he does not attempt to form a separate expectation for each possible contingency). And the rationale for doing this is to make learning easier.

[8]These are assumed to be rightly perceived by the players.

[9]As for sequential equilibria, an analogy-based expectation equilibrium always exists in finite environments.

[10]The purpose here is to illustrate a few phenomena caused by analogy reasoning. In so doing (and partly for pedagogical reasons), we will rely on highly simplified game representations for which -one may argue- the approach was not originally designed.

[11]Clearly, this is not so if this player plays against Nature or if other players have a dominant strategy. Then a coarse partitioning has the sole effect of making this player 's choice of strategy possibly suboptimal without affecting the behaviors of others. But, otherwise, a coarse partitioning of, say, player i may well induce (in equilibrium) a change of strategies of players other than i (as a response to a change of strategy of player i). When such a change of strategies is good for player i, player i may end up with a strictly higher payoff.

why one cannot interpret an analogy-based expectation equilibrium as an equilibrium of a modified game with the same payoff and move structures, but possibly with a different information structure.

We next apply the analogy-based expectation approach to the so called finite horizon paradoxes. We show both in the centipede game and in the finitely repeated prisoner's dilemma that, for some analogy partitioning, there are equilibria in which there is a fair amount of cooperation throughout the game except possibly toward the end of the game at which time some opportunistic behavior must occur. (The last phase echoes a phenomenon referred to as the end effect in the experimental literature, see Selten and Stoecker 1986, but also McKelvey-Palfrey 1992 and Nagel-Tang 1998.)

The claim has already been illustrated above for the finitely repeated prisoner's dilemma in which players use two analogy classes according to whether or not some opportunistic behavior was previously observed. The key reason why the logic of backward induction fails in this case is that players do not perceive exactly *when* the other player starts having an opportunistic behavior. Due to their analogy partitioning, players only have a fuzzy perception of their opponents' behavior, and they play cooperatively most of the time because by so doing (and only by so doing) they expect the other player to keep playing cooperatively with a large probability.[12]

Our next application deals with ultimatum and take-it-or-leave-it offer games. Suppose that the proposer can make any possible offer, but that he has expectations about the acceptance probability of the responder only according to whether his offer is above or below a threshold (i.e., whether or not his offer is generous). We show that the responder may get a payoff that lies strictly above his reservation utility (i.e. his payoff from refusing any agreement).

To see the point, assume the proposer is to make an offer that lies in a given (analogy) class. He will always pick the least generous offers among these. This is because (due to his analogy partitioning) he has the same (acceptance) expectation for all such offers, and the least generous offer among these is clearly the one he likes best given such an expectation. So analogy grouping has the effect of endogenously discretizing the action space of the proposer (to the lower extreme points of his analogy classes), which in turn explains why the responder need not be stuck to his reservation utility.[13]

[12]The line of argument is completely different from that developed in the crazy type approach (Kreps et al 1982). While the current approach takes the view that agents are less sophisticated than in the standard paradigm, the crazy type approach assumes a great sophistication on the part of the players (in particular, players make perfect inferences from observed behaviors onto the likelihood of their opponents' types).

[13]In a bargaining game a la Rubinstein, such a paradigm might explain why an agreement need not be

In the last part of the paper, we provide some general discussion. We expand on the interpretation of the solution concept in terms of the information treatment of the players at the learning stage. We differentiate our insights about the finite horizon paradox from those obtained when players have imperfect recall. And, we suggest two principles that may help structure analogy partitioning in future research. A discussion of the literature and concluding remarks appear in the last section. Missing proofs can be found in the Appendix.

2 A general framework

2.1 The class of games

We consider multi-stage games with almost perfect information and perfect recall (see Fudenberg and Tirole 1991 section 3.3.2). That is, simultaneous moves and moves by Nature are allowed. But, in any stage, all previous moves are assumed to be known to every player.

In the main part of the paper, we will restrict attention to games with a finite number of stages such that, at every stage and for every player (including Nature), the set of pure actions is finite. This class of (finite) multi-stage games with almost perfect information is referred to as Γ.

The standard representation of an extensive form game in class Γ includes the set of players $i = 1, ...n$ denoted by N, the game tree Υ, and the preferences $\succsim_i$ of every player i over outcomes in the game.

A node in the game tree Υ is denoted by h, and the set of nodes is denoted by H. The set of nodes at which player i must move is denoted by H_i, and for every such node $h \in H_i$, we let $A_i(h)$ denote player i's action space at node h.

Remark: When interpreting experiments, it may be meaningful to view the players as being engaged in a variety of games as opposed to only one game. One can represent this as a metagame made of an extra move by Nature in stage 0 which would determine the effective game to be played (according to the frequency with which each (original) game was played).

Classes of analogy:

Each player i forms an expectation about the behavior of other players by pooling together several contingencies in which these other players must move. Each such *pool* of contingencies is referred to as a *class of analogy*. And player i forms an expectation about the *average* behavior in each analogy class that he considers.

reached immediately.

Formally, each player i partitions the set $\{(j,h) \in N \times H_j,\ j \neq i\}$ into subsets α_i referred to as analogy classes.[14] The collection of player i's analogy classes α_i is referred to as player i's analogy partition, and it is denoted by An_i. When (j,h) and (j',h') are in the same analogy class α_i, we require that $A_j(h) = A_{j'}(h')$. That is, in two contingencies (j,h) and (j',h') that player i treats by analogy, the action space of the involved player(s) should be the same.[15] The common action space in the analogy class α_i will be denoted by $A(\alpha_i)$. The profile of analogy partitions $(An_i)_{i\in N}$ will be denoted by An.

Strategic environment:

A strategic environment in our setup not only specifies the set of players N, the game tree Υ and players' preferences $\succsim_i$. It also specifies how the various players partition the set of nodes at which other players must move into classes of analogy, which is summarized in An. A strategic environment is thus formally given by $(N, \Upsilon, \succsim_i, An)$.

2.2 Concepts

Analogy-based expectations:

An analogy-based expectation for player i is denoted by β_i. It specifies for every player i's analogy class α_i, a probability measure over the action space $A(\alpha_i)$. This probability measure is denoted by $\beta_i(\alpha_i)$, and $\beta_i(\alpha_i)$ should be interpreted as player i's expectation about the average behavior in class α_i.

Strategy:

A behavior strategy for player i is denoted by σ_i. It is a mapping that assigns to each node $h \in H_i$ at which player i must move a distribution over player i's action space at that node.[16] That is, it specifies for every $h \in H_i$ a distribution - denoted $\sigma_i(h) \in \Delta A_i(h)$ - according to which player i selects actions in $A_i(h)$ when at node h. We let σ_{-i} denote the strategy profile of players other than i, and we let σ denote the strategy profile of all players.

Sequential rationality:

Given his analogy-based expectation β_i, player i constructs a strategy profile for players other than i that assigns player j to play according to $\beta_i(\alpha_i)$ at node h whenever $(j,h) \in \alpha_i$.

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[14] A partition of a set X is a collection of subsets $x_k \subseteq X$ such that $\bigcup_k x_k = X$ and $x_k \cap x_{k'} = \emptyset$ for $k \neq k'$.

[15] More generally, we could allow the players to relabel the original actions of the various players as they wish. From that prespective, $A_j(h)$ should only be required to be in bijection with $A_{j'}(h')$ (as opposed to being equal).

[16] Mixed strategies and behavior strategies are equivalent since we consider games of perfect recall.

(This is the most natural strategy profile compatible with player i's belief β_i.) The criterion used by player i is that of best-response against this induced strategy profile at every node where player i must move.

More precisely, for every β_i and $j \neq i$, we define the β_i-perceived strategy of player j, $\sigma_j^{\beta_i}$, as

$$\sigma_j^{\beta_i}(h) = \beta_i(\alpha_i) \text{ whenever } (j,h) \in \alpha_i.$$

Given player i's strategy σ_i and given node h, we let $\sigma_i \mid_h$ denote the continuation strategy of player i induced by σ_i from node h onwards. Similarly, we let $\sigma_{-i} \mid_h$ and $\sigma \mid_h$ be the strategy profiles induced by σ_{-i} and σ, respectively, from node h onwards. We also let $u_i^h(\sigma_i \mid_h, \sigma_{-i} \mid_h)$ denote the expected payoff obtained by player i when the play has reached node h, and players behave according to the strategy profile σ.

Definition 1 *(Criterion) Player i's strategy σ_i is a sequential best-response to the analogy-based expectation β_i if and only if for all strategies σ_i' and all nodes $h \in H_i$,*

$$u_i^h(\sigma_i \mid_h, \sigma_{-i}^{\beta_i} \mid_h) \geq u_i^h(\sigma_i' \mid_h, \sigma_{-i}^{\beta_i} \mid_h).$$

Consistency:

In equilibrium, we require the analogy-based expectations of the players to be *consistent.* That is, to correspond to the real average behavior in every considered class where the weight given to the various elements of an analogy class must itself be consistent with the real probabilities of visits of these various elements.

We think of the consistency requirement as resulting from a learning process in which players would eventually manage to have correct analogy-based expectations. In line with the literature on learning in games (see Fudenberg-Levine 1998), we distinguish according to whether or not consistency is only required for those analogy classes that are reached with strictly positive probability.[17]

To present formally the consistency idea, we denote by $P^\sigma(h)$ the probability that node h is reached according to the strategy profile σ.

Definition 2 *(Weak Consistency) Player i's analogy based expectation β_i is consistent with the strategy profile σ if and only if for all $\alpha_i \in An_i$:*

$$\beta_i(\alpha_i) = \left(\sum_{(j,h)\in\alpha_i} P^\sigma(h) \cdot \sigma_j(h) \right) / \left(\sum_{(j,h)\in\alpha_i} P^\sigma(h) \right) \tag{1}$$

whenever $P^\sigma(h) > 0$ for some h and j such that $(j,h) \in \alpha_i$.

[17] When it is required for unreached classes, the underlying learning model should involve some form of trembling (or exogenous experimentation). When it is not, trembles are not necessary.

This definition deserves a few comments. Suppose that players repeatedly act in the environment as described above. Suppose further that the true pattern of behavior adopted by the players is that described by the strategy profile σ. Consider player i who tries to forecast the average behavior in the analogy class α_i, assumed to be reached with positive probability (according to σ).

The actual behavior in the analogy class α_i is an average of what every player j actually does in each of the nodes h where $(j,h) \in \alpha_i$, that is, $\sigma_j(h)$. The correct weighting of $\sigma_j(h)$ should coincide with the frequency with which (j,h) is visited (according to σ) relative to other elements in α_i. The correct weighting of $\sigma_j(h)$ should thus be $P^{\sigma}(h)/\left(\sum_{(j,h)\in\alpha_i} P^{\sigma}(h)\right)$, which in turn yields expression (1).

It should be noted that Definition 2 places no restrictions on player i's expectations about those analogy classes that are not reached according to σ. The next definition proposes a stronger notion of consistency (in the spirit of trembling hand or sequential equilibrium) that places restrictions also on those expectations.

Formally, we define Σ^0 to be the set of totally mixed strategy profiles, i.e. strategy profiles σ such that for every player j, for every node $h \in H_j$ at which player j must move, any action a_j in the action space $A_j(h)$ is played with strictly positive probability. For every strategy profile $\sigma \in \Sigma^0$, all analogy classes are reached with positive probability. Thus, there is a unique analogy-based expectation β_i that is *consistent* with σ in the sense of satisfying condition (1) for all analogy classes α_i. Denote this analogy-based expectation by $\beta_i \langle \sigma \rangle$.

Definition 3 *(Strong consistency) Player i's analogy-based expectation β_i is strongly consistent with σ if and only if there exists a sequence of totally mixed strategy profiles $\left(\sigma^k\right)_{k=1}^{\infty}$ that converges to σ such that the sequence $\left(\beta_i \ \sigma^k\rangle\right)_{k=1}^{\infty}$ converges to β_i.*

Solution concept:

In equilibrium, we require that at every node players play best-responses to their analogy-based expectations (sequential rationality) and that expectations are consistent. We define two solution concepts according to whether or not consistency is imposed for analogy classes that are not reached along the played path. And we refer to a pair (σ, β) of strategy profile and analogy-based expectation profile as an *assessment.*

Definition 4 *An assessment (σ, β) is an Analogy-Based Expectation Equilibrium (resp. a Self-Confirming Analogy-Based Expectation Equilibrium) if and only if for every player $i \in N$,*

1. *σ_i is a* **sequential best-response** *to β_i and*
2. *β_i is* **strongly consistent (resp. consistent)** *with σ.*

At this point, it may be worth stressing a few notable differences between an analogy-based expectation equilibrium and a sequential equilibrium of an extensive form game with incomplete information. First, observe that an analogy partition of, say player i, is a partition of the nodes where players *other than* i must move. It is thus of a different nature than player i's information structure in a game with incomplete information which refers to a partition of the nodes where player i *himself* must move.[18] Second, observe the different nature of player i's analogy-based expectation $\beta_i(\cdot)$ and of player i's belief system in extensive form games with incomplete information. Here $\beta_i(\alpha_i)$ is an expectation (or belief) about the *average behavior* of players other than i in class α_i. It is *not* a belief, say, about the likelihood of the various elements (j, h) pooled in α_i. Finally, note that in the analogy setup, the same *expectation* is used to assess the behavior of the opponent(s) in two elements of an analogy class. By contrast, in the incomplete information setup a player *behaves* in the same way at two nodes of a common information set. We will illustrate further through examples the differences between the two solution concepts, thus revealing that the analogy-based expectation equilibrium is an entirely new solution concept.

We conclude this general presentation by making a few remarks:

1. To the extent that the number of analogy classes α_i considered by player i is small, player i has few features of other players' behavior to learn, which makes the consistency requirement more plausible from a learning perspective than in the perfect rationality paradigm.

2. A priori there are strategies other than $\sigma_{-i}^{\beta_i}$ that could generate the analogy-based expectation β_i. A more elaborate criterion than the one considered in Definition 1 would view player i as playing a best-response against *some* strategy profile σ'_{-i} compatible with β_i but not necessarily $\sigma_{-i}^{\beta_i}$. The corresponding solution concepts would be somewhat more complicated to present (but most of the insights developed below would continue to hold for such alternative specifications).

3. Our setup assumes that every (j, h), $h \in H_j$ belongs to one analogy class α_i and only one (An_i is a partition of (j, h)). In some applications, it may be meaningful to assume that every (j, h) may be assigned to various analogy classes depending on the realization of some random device.[19] Such a modification would allow us to have a more continuous representation of *similarity* than the current model permits. (That is, depending on

[18](with the requirement that player i's action spaces at two nodes of a common information set are equal)

[19]The realization is to be interpreted as how the player views analogies at the time he is engaged in a specific interaction (and these views may vary stochastically during and after the learning phase).

the distributions of assignments to analogy classes, two elements (j, h) and (j', h') may have various degrees of similarity.) Extending the model in this direction would be somewhat cumbersome,[20] but we believe most of the insights developed below would continue to hold.

4. We have assumed that player i's analogy classes are partitions of the nodes where players *other than* i must move. In some cases, it may be meaningful to allow the players to predict the behavior of their opponents also based on their *own* behavior. There is no difficulty with allowing the analogy classes α_i to also include (i, h) such that at node h player i must choose an action in $A(\alpha_i)$ (the same action space as the one faced by the other players involved in α_i).[21]

5. The setup could easily be extended to cover the case where players have private information. However, this would significantly complicate the description of the setup. For expositional (rather than conceptual) reasons, we have chosen to focus on games with almost perfect information.

2.3 Preliminary results

Two simple observations follow. The first one shows the relation of the analogy-based expectation equilibrium to subgame perfection when all players use the finest partitioning as their analogy device. The second one shows the existence of analogy-based expectation equilibria in finite environments.

Proposition 1 *Consider an environment* $(N, \Upsilon, \succsim_i, An)$ *in which all players use the finest analogy partitioning.*[22] *Then if* (σ, β) *is an analogy-based expectation equilibrium of* $(N, \Upsilon, \succsim_i, An)$*,* σ *is a Subgame Perfect Nash Equilibrium of* $(N, \Upsilon, \succsim_i)$*.*

Proof. When players use the finest analogy partitioning, strong consistency of β_i with respect to σ implies that $\sigma_{-i}^{\beta_i} = \sigma_{-i}$. Proposition 1 then follows from Definition 1. ■

Remark: When at least one player, say player i, does not use the finest partition as his analogy device, the play of an analogy-based expectation equilibrium need not correspond to

[20] The notion of best-response σ_i to an analogy-based expectation β_i should then be thought of as a stochastic one, for each possible assignement of (j, h) to α_i. The notion of consistency should also be modified to take into account the frequency with which an element (j, h) is assigned to α_i.

[21] However, it should be understood that the corresponding analogy-based expectation $\beta_i(\alpha_i)$ is used by player i only to construct a strategy profile for players other than i (see Definition 1).

[22] We say that all players use the *finest* analogy partitioning if there are no i, (j, h), $(j', h') \neq (j, h)$ and $\alpha_i \in An_i$ such that $(j, h) \in \alpha_i$ *and* $(j', h') \in \alpha_i$.

that of a Subgame Perfect Nash Equilibrium. This is because in an analogy-based expectation equilibrium (σ, β), player i's strategy σ_i is required to be a best-response to $\sigma_{-i}^{\beta_i}$. But, $\sigma_{-i}^{\beta_i}$ need not (in general) coincide with σ_{-i} as in a Subgame Perfect Nash equilibrium. This will be further illustrated throughout the paper.

Proposition 2 *(Existence) Every finite environment* $(N, \Upsilon, \succsim_i, An)$ *has at least one analogy-based expectation equilibrium.*

Proof. The strategy of proof is the same as that for the existence proof of sequential equilibria (Kreps and Wilson 1982). We mention the argument, but for space reasons we do not give the details of it. First, assume that in every node $h \in H_i$, player i must choose every action $a_i \in A_i(h)$ with probability no smaller than ε (this is in spirit of Selten 1975).[23] It is clear than an analogy-based expectation equilibrium with such additional constraints must exist. Call $(\sigma^\varepsilon, \beta^\varepsilon)$ one such profile of strategies and analogy-based expectations. By compactness properties (which hold in the finite environment case), some subsequence must be converging to say (σ, β), which is an analogy-based expectation equilibrium. ■

3 Various effects of analogy reasoning

3.1 Analogy reasoning can be good or bad

Bundling contingencies by analogy can either benefit or hurt a player. For the sake of illustration, consider the following (simplistic) environment. Two normal form games G and G' are being played in parallel. Game G is played with probability ν and game G' is played with probability $1 - \nu$. (In the formulation of Section 2, the game tree Υ consists of a first move by Nature about the selection of the game - according to the probabilities ν and $1 - \nu$ - then followed by the normal form game G or G' accordingly.) There are two players $i = 1, 2$ in G and G'. In both G and G', player i must choose an action a_i in the *same* finite action space A_i.

In the game tree Υ, a node can be identified with a normal form game G or G'. We assume that player 2 uses the finest partitioning (i.e., player 2 uses two analogy classes $\{(1, G)\}$ and $\{(1, G')\}$).

We wish to compare the equilibrium payoff obtained by player 1 in each of the subgames G, G' according to whether player 1 uses the *finest* partitioning or the *coarsest* partitioning.

Clearly, if player 2 has a dominant strategy in both games G and G', player 1's equilibrium payoff - in both G and G' - is no smaller when player 1 uses the finest partitioning as opposed

[23] This requires amending Definition 1 to incorportate such constraints in the maximization programmes.

to the coarsest partitioning.

This is because whatever the partitioning of player 1, player 2 will in equilibrium select his dominant strategy in both G and G'. And the finest partitioning of player 1 allows player 1 to pick a best-response to player 2's dominant strategy in both G and G', which results in the highest payoff player 1 can hope to get (in both G and G') given player 2's behavior.

When player 2 has no dominant strategy, however, analogy reasoning may benefit player 1, as the following example shows.

Example 1: Consider the following situation

	L	M	R
U	$5,1$	$0,1$	$2,2$
D	$3,1$	$3,0$	$1,0$

Game G

	L	M	R
U	$3,0$	$3,1$	$1,0$
D	$0,1$	$5,1$	$2,2$

Game G'

where in each cell the left and right numbers indicate players 1 and 2' payoffs, respectively. Both games are assumed to be played with equal probability, i.e. $\nu = \frac{1}{2}$. In both G and G', the action space of players 1 and 2 are $A_1 = \{U, D\}$ and $A_2 = \{L, M, R\}$, respectively.

The example is such that both G and G' have a unique Nash equilibrium, which is UR in game G and DR in game G'. Thus, when both players use the finest partitioning, player 1 gets a payoff of 2 in both subgames.

Suppose now that player 1 uses the coarsest partitioning (while player 2 uses the finest). The following assessment is an analogy-based expectation equilibrium.

Strategy profile: Player 1 plays D in game G and U in game G'. Player 2 plays L in game G and M in game G'.

Analogy-based expectations: Player 1 expects player 2 to play L and M each with probability $\frac{1}{2}$ (in his unique analogy class $\{(2, G), (2, G')\}$). Player 2 expects player 1 to play D in game G and U in game G'.

To check that the above assessment is an equilibrium, note that given the strategy profile, players' analogy-based expectations are consistent. Then given player 1's analogy-based expectation, player 1 chooses D (resp. U) rather than U (resp. D) in game G (resp. G') because $\frac{1}{2}(3+3) > \frac{1}{2}(0+5)$. Given player 1's strategy, player 2's best-response is L in game G and M in game G'.

Finally, note that according to the above strategy profile player 1 gets a payoff of 3 in both G and G', which is strictly larger than 2 - the equilibrium payoff obtained by player 1 when he uses the finest partitioning. ■

The key feature of Example 1 is that player 2 does not play in the same way when player 1 uses the finest partitioning and when he uses the coarsest partitioning. The coarseness of player 1's partitioning does induce player 1 not to optimize against player 2's actual behavior in G and G' (because the best-response would be U (and not D) in game G and D (and not U) in game G'). However, it allows player 1 to find it optimal to play D (resp. U) in game G (resp. G'), which in turn induces player 2 to play an action that is more favorable to player 1.

Comment : In the analogy-based expectation equilibrium shown in Example 1, both players 1 and 2 behave differently in games G and G'. Thus, (even by varying the payoff matrix specification) it is not possible to interpret their behavior as resulting from a lack of information as to which game (G or G') is being played.

3.2 The Centipede game

Consider the centipede game CP_K depicted in Figure 1 (see Rosenthal 1982).

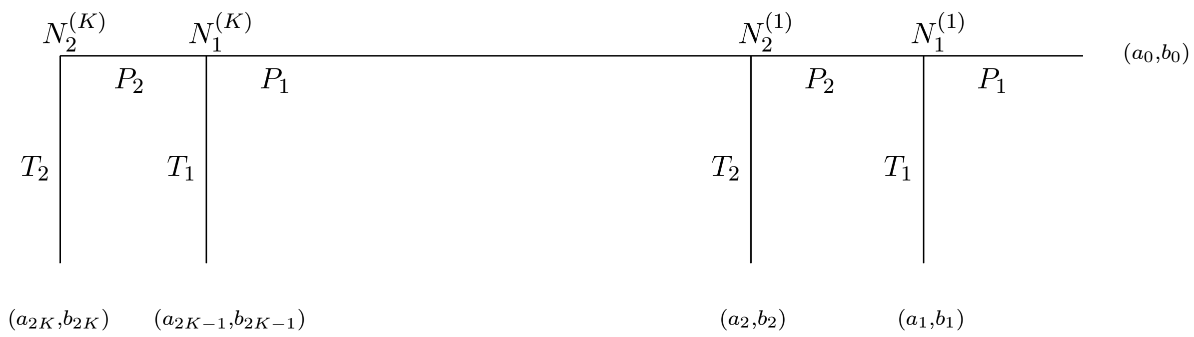

Figure 1: The centipede game

Two players $i = 1, 2$ move in alternate order starting with player 2. In each period, the player whose turn it is to move, say player i, may either *Take* or *Pass*, i.e. $A_i = \{Pass, Take\}$. If he Takes, this is the end of the game. If he Passes, the game proceeds to the next stage where it is the other player's turn to move unless the game has reached the last period $2K$ in which case this is the end of the game. Nodes at which player 1 must move are labelled $N_1^{(k)}$, $k = 1, ...K$ where $N_1^{(1)}$ designates the last node (i.e., period $2K$) at which player 1 must move. Similarly, nodes at which player 2 must move are labelled $N_2^{(k)}$, $k = 1, ...K$ where $N_2^{(1)}$ designates the last node (in period $2K - 1$) at which player 2 must move. As shown in Figure 1, if player 2 Takes at node $N_2^{(k)}$, the payoffs to players 1 and 2 are a_{2k} and b_{2k}, respectively. If player 1 Takes at node $N_1^{(k)}$, the payoffs to players 1 and 2 are a_{2k-1} and b_{2k-1}, respectively. If player 1 Passes at node $N_1^{(1)}$, the payoffs to players 1 and 2 are a_0 and b_0, respectively. All a_t and b_t, $t = 0, ...2K$ are assumed to be integers that satisfy for all $k \geq 1$:

$$\begin{aligned} a_{2k-1} &> a_{2k-2} > a_{2k+1} > a_{2k} \\ b_{2k-2} &> b_{2k-3} > b_{2k} > b_{2k-1} \end{aligned} \tag{2}$$

These conditions guarantee that 1) the unique Subgame Perfect Nash Equilibrium (SPNE) of CP_K is such that player 2 Takes in the first period (this follows from $a_{2k+1} > a_{2k}$ and $b_{2k} > b_{2k-1}$), and 2) in any period $t \leq 2K - 2$, both players are better off if *Take* occurs two periods later, i.e. in period $t + 2$, than if it occurs in the current period t (this follows from $a_t > a_{t+2}$ and $b_t > b_{t+2}$ for all $t \leq 2K - 2$).

The prediction of the SPNE sounds relatively unintuitive, especially when the number of periods $2K$ is large. As we now illustrate, the analogy approach explains why players may Pass most of the time in the centipede game, at least for long enough versions of the game.

In order to deal with the effect of increasing the number of periods in CP_K, we will consider infinite sequences of integers $(a_k)_{k=0}^{\infty}$, $(b_k)_{k=0}^{\infty}$ satisfying (2). We will also assume that the difference between two consecutive payoffs is uniformly bounded from above. That is, there exists $\Delta > 0$ such that for all $t \geq 0$,

$$|a_t - a_{t+1}| < \Delta \text{ and } |b_t - b_{t+1}| < \Delta. \tag{3}$$

Regarding analogy partitioning, we will mostly focus on the coarsest partitioning. That is, each player i pools together *all* the nodes $N_j^{(k)}$ at which player j, $j \neq i$ must move into a single class of analogy α_i:

$$\alpha_i = \left\{(j, N_j^{(k)}), 1 \leq k \leq K\right\}.$$

The strategic environment is thus described by the set of players $N = \{1, 2\}$, the game tree CP_K, players' preferences $\succsim_i$ as defined by a_t, b_t, and the analogy partitioning structure An as described by α_1 and α_2: $(N, CP_K, \succsim_i, An)$.

The following Proposition characterizes the equilibria that employ pure strategies whenever

$$\frac{K-1}{K} b_{2k} + \frac{1}{K} b_{2k+1} > b_{2k+2} \text{ for all } k \geq 0. \tag{4}$$

Proposition 3 *Suppose that condition (4) holds and consider the environment $(N, CP_K \succsim_i, An)$. There are two possible equilibrium paths corresponding to self-confirming analogy-based expectation equilibria in pure strategies: Either player 2 Takes in the first period or the game reaches period $2K$ in which player 1 Takes.*

Proof. Note first that player i's analogy-based expectation β_i reduces here to a single probability measure $\beta_i(\alpha_i) = \lambda^i \cdot Pass + (1-\lambda^i) \cdot Take \in \Delta A_j$, which stands for player i's expectation about the average behavior of player j throughout the game. In the Appendix we check that the two mentioned outcomes can be obtained. To show that there are no other possible outcomes in any pure strategy self-confirming analogy-based expectation equilibrium, consider the outcome in which player i Takes at node $N_i^{(k)}$, and $N_i^{(k)}$ differs from $N_2^{(K)}$ and $N_1^{(1)}$.

If a pure strategy analogy-based expectation equilibrium leads to that outcome, it must be (by consistency) that player i's analogy based expectation satisfies $\beta_i(\alpha_i) = Pass$, since on the equilibrium path, player j would always Pass. Player i's best response to such a β_i depends on whether $i = 1$ or 2. If $i = 1$, player 1's best response to $\beta_1(\alpha_1) = Pass$ is to Take at node $N_1^{(1)}$ (but then the outcome would be that player 1 Takes at node $N_1^{(1)}$, which we have ruled out). If $i = 2$, player 2's best response to $\beta_2(\alpha_2) = Pass$ is to Pass always, which is in contradiction with player 2 Taking at node $N_2^{(k)}$.

Finally, the outcome in which both players Pass in every period cannot be an analogy-based expectation equilibrium outcome because whatever player 1's expectation, player 1 strictly prefers Taking at node $N_1^{(1)}$ to Passing always. ■

Proposition 3 leaves open what happens when condition (4) does not hold. And it does not deal with equilibria in mixed strategies. The next Proposition provides the main missing information (still assuming that conditions (2) and (3) hold):[24]

Proposition 4 *There exists an integer $\overline{m}$ such that for all $K > \overline{m}$: (1) $(N, CP_K, \succsim_i, An)$ has an analogy-based expectation equilibrium (σ, β) in which each player i Passes with probability*

[24] There are two equilibria in addition to those reviewed in Proposition 4: the Subgame Perfect Nash Equilibrium, and a mixed strategy equilibrium in which Take may occur in the first two periods (it is such that each player $i = 1, 2$ plays in mixed strategies in $N_i^{(K)}$and Takes with probability 1 in all other nodes).

A slight modification in which all equilibria are of the form depicted in Proposition 4 is as follows. Let K be an odd number larger than 2 so that $a_0 > a_K > a_{2K}$ and $b_0 > b_K > b_{2K}$. Players play game CP_K with probability $\nu_{CP} > 0$ but also game F with probability $\nu_F > 0$. Game F has two nodes and it corresponds to CP_K when player 2 first decides whether or not to Pass to the middle of the game and then player 1 decides whether or not to go to the end. More precisely, F starts with node M_2 where player 2 must choose between $Pass$ and $Take$. If player 2 Takes, players 1 and 2' payoffs are a_{2K} and b_{2K}, respectively. If player 2 Passes, the game moves to node M_1 where player 1 must choose between $Pass$ and $Take$. If player 1 Takes, the payoffs to players 1 and 2 are a_K and b_K, respectively; if he Passes, this is the end of the game and the payoffs to players 1 and 2 are a_0 and b_0, respectively.

Whenever each player i uses a single class of analogy, players must Pass in most periods of CP_K when K is large enough. The point is that, in game F, Passing is a dominant strategy for both players. This forces the analogy-based expectation to put some significant weight (i.e., no smaller than ν_F) on Pass, which in turn leads the players to Pass most of the time in CP_K.

1 *in the first* $K - \overline{m}$ *nodes. (2) Any self-confirming analogy-based expectation equilibrium in which each player* i *Passes with probability* 1 *in the first node* $N_i^{(K)}$ *is such that each player* i *Passes with probability* 1 *in the first* $K - \overline{m}$ *nodes.*

Proposition 4 (1) shows that irrespective of the length $2K$ of the game, there is an equilibrium (possibly in mixed strategies) in which Take can only occur in a finite number of periods toward the end of the game.[25] Proposition 4 (2) shows that there cannot be equilibria in which Take occurs in the middle phase of the game (i.e. between period 3 and period $2K - 2\overline{m}$). Thus, restricting attention to equilibria in which Take does not occur in the first two periods, the end phase (in which Take may occur) can never last more than $2\overline{m}$ periods, irrespective of the total duration $2K$ of CP_K.

Comment: In the above analysis of the centipede game CP_K, we have assumed that players use the coarsest analogy partitioning. In many instances though players may recognize that the last few moves are very different from the first few and from the middle ones, thus suggesting that players would use three analogy classes rather than one. In this three analogy class setting, there is still an equilibrium in which players Pass most of the time in long enough CP_K. In the last few nodes, players may Take. But, in the middle nodes, the same logic as the one developed above applies (thus players Passing most of the time is a plausible outcome even in this three class setting).[26]

3.3 Finitely Repeated Prisoner's Dilemma

Consider the Prisoner's Dilemma PD whose payoff matrix is given by:

	D	C
D	$0, 0$	$1 + g_1, -l_2$
C	$-l_1, 1 + g_2$	$1, 1$

Game PD

with l_i, $g_i > 0$ for $i = 1, 2$, and each player $i = 1, 2$ has to choose simultaneously an action in $\{D, C\}$.

We consider a T repetition of such games where the values of g_i may change from period to period. That is, in each period there is a draw by Nature that determines the value of g_i, $i = 1, 2$, for the current period (and, for simplicity, the values of l_i are assumed to remain constant throughout the game).

[25] When condition (4) does not hold, this may involve an equilibrium in mixed strategies.

[26] The same conclusion would also hold if the thresholds separating the three classes were randomly determined (see Remark 3 at the end of Section 2.2).

To fix ideas, we assume that the distributions of g_i are independent from period to period and across players, and that in each period, g_i takes value $\underline{g}$ with probability $\underline{\nu}$ and $\overline{g}$ with probability $\overline{\nu}$ where $\overline{\nu} + \underline{\nu} = 1$ and $\overline{g} > \underline{g} > 0$.[27]

We denote by $z^{(t)}$ the joint draw of (g_1, g_2) in period t. Players are assumed to be risk neutral and they do not discount future payoffs. We denote by PD_s the associated game tree, and by $\succsim_i$ player i's preferences.

Nodes in PD_s correspond to histories of length 0 to T specifying the action profiles played in earlier periods (if any) and the draws by Nature in all periods up to (and including) the current period. That is, the history of length 0 is $z^{(1)}$ specifying the draws g_1 and g_2 for the first period. A history h of length $t > 0$ is $((a^{(1)}, z^{(1)}); ...; (a^{(t)}, z^{(t)}); z^{(t+1)})$ where $a^{(k)} = \left(a_1^{(k)}, a_2^{(k)}\right)$, $z^{(k)} = \left(g_1^{(k)}, g_2^{(k)}\right)$ and $a_i^{(k)} \in \{D, C\}$ stands for the action played by player i in period k while $g_i^{(k)} \in \left\{\underline{g}, \overline{g}\right\}$ stands for the draw of g_i in period k.

Each player i partitions the set of (j, h) into two classes:[28]

$$\begin{aligned} \alpha_i &= \{(j,h) \mid h \text{ contains no } D\} \\ \alpha_i' &= \{(j,h) \mid h \text{ contains at least one } D\} \end{aligned}$$

We define $u_T = 1 + (\underline{\nu} \cdot \underline{g} + \overline{\nu} \cdot \overline{g})$, and the sequence $(u_t)_{t<T}$ recursively by[29]

$$u_t = 1 + (\underline{\nu} \cdot u_{t+1} + \overline{\nu} \cdot \overline{g}).$$

We assume that $u_T < \overline{g}$ and that no u_t in this sequence is equal to $\overline{g}$. We define $\overline{m}$ as the integer such that $u_{T-\overline{m}+1} < \overline{g} < u_{T-\overline{m}}$.

Proposition 5 *For T large enough, the following strategy profile is part of an analogy-based expectation equilibrium of $(N, PD_s, \succsim_i, An)$: For each player i, play D if one (or more) D occurred so far; Otherwise, in all periods t, $t < T - \overline{m}$, play C; in all periods t, $T - \overline{m} < t < T$, play C if $g_i^{(t)} = \underline{g}$ and D if $g_i^{(t)} = \overline{g}$; in period T, play D.*

The logic of the equilibrium is as follows. Players rightly perceive that in class α_i', i.e. if some D was played earlier, only D can be expected next. In class α_i, player j chooses C most of the time except toward reaching the end of the game: player i' expectation is thus close to

[27]To keep in line with the class of games considered in Section 2, we assume that the draws of both g_1 and g_2 are immediately revealed to both players. However, this is immaterial for the analysis below.

[28]History $h = ((a^{(1)}, z^{(1)}); ...; (a^{(t)}, z^{(t)}); z^{(t+1)})$ is said to contain at least one D if there exist $i = 1, 2$ and $k \leq t$ such that $a_i^{(k)} = D$. It is said to contain no D otherwise.

[29]u_t stands for the expected payoff of player i at date $t-1$ when no C previously occurred and player i anticipates that (1) he will play D in the next period if $g_i = \overline{g}$ (or if $t = T$) and that (2) player j plays C if no D previously occurred and D otherwise.

C in this class. Given such an expectation, player i considers breaking the sequence of C (by playing D) only when the immediate gain g_i from playing D is not too small relative to the loss incurred by triggering a D sequence. This occurs only toward the end of the game (i.e. in the last $\overline{m}+1$ periods) when the draw of g_i is $\overline{g}$ (and also in the last period irrespective of the realization of $g_i^{(T)}$).

3.4 Ultimatum and Take-It-Or-Leave-It games

Consider the following Take-it-or-Leave-it model. There are two players $i = 1, 2$ and a pie of size 1. Player 1 makes a partition offer $(x, 1-x)$, $x \in [0,1]$ to player 2 who may either accept or reject it.[30] If he accepts, players 1 and 2 get x and $1-x$, respectively. If player 2 rejects the offer, player 1 gets 0 and player 2 gets an outside option payoff equal to v^{out}, where $0 < v^{out} < 1$. Denote by $N = \{1, 2\}$ the set of players, $\succsim_i$ player i's preferences, and TL the game tree associated with the above setup.

Standard analysis suggests that player 1 will propose $(1-v^{out}, v^{out})$ and that player 2 will accept it. When player 1 forms his expectation about player 2's probability of acceptance by analogy, we now show that it may well be that either player 1 makes a much more generous offer than v^{out} to player 2 or that player 1 makes an offer that is rejected by player 2 depending on the partitioning.

Specifically, a node in TL at which player 2 must move can be identified with x where $(x, 1-x)$ is the offer made by player 1. We assume that player 1 partitions the set of $(2, x)$ into two classes:[31]

$$\begin{aligned} \alpha_1^{low} &= \{(2,x) \mid \overline{x} < x \leq 1\} \\ \alpha_1^{high} &= \{(2,x) \mid 0 \leq x \leq \overline{x}\} \end{aligned}$$

where α_1^{low} (resp. α_1^{high}) corresponds to the class of outrageous (resp. generous) offers.

Proposition 6 *(1) When $1-\overline{x} < v^{out}$, any analogy-based expectation equilibrium is such that there is no agreement: player 1 gets 0, player 2 gets v^{out}. (2) When $v^{out} < 1-\overline{x}$, there is a unique analogy-based expectation equilibrium: player 1 proposes $(\overline{x}, 1-\overline{x})$, and player 2 accepts.*

[30] The action space of player 1 in this example is continuous (which is not covered by the framework of Section 2). The analysis presented below can be viewed as corresponding to the limit of the finite grid case as the grid becomes finer and finer. (Alternatively, one can extend the definitions of consistency and of analogy-based expectation equilibrium for this specific example.)

[31] The intervals are closed as indicated to guarantee the existence of an equilibrium.

Proof. The analogy-based expectation of player 1 is of the form $\beta_1(\alpha_1^r) = \lambda^r \cdot$'Accepts'$+(1-\lambda^r)\cdot$'Rejects' with $r = low, high$. If $\lambda^{high} > 0$ (resp. $\lambda^{low} > 0$), player 1's best-response to β_1 cannot be to offer $(x, 1-x)$ with $x < \overline{x}$ (resp. $\overline{x} < x < 1$). (1) When $1 - \overline{x} < v^{out}$, neither $(1, 0)$ nor $(\overline{x}, 1 - \overline{x})$ are acceptable by player 2. Only a disagreement can occur. (2) When $v^{out} < 1 - \overline{x}$, $\lambda^{high} = 1$, $\lambda^{low} = 0$, player 1 proposing $(\overline{x}, 1 - \overline{x})$ and player 2 accepting any offer $(x, 1 - x)$ with $1 - x \geq v^{out}$ gives rise to an analogy-based expectation equilibrium. It is also easy to see that there is no other analogy-based expectation equilibrium. (For example, a disagreement cannot be part of an equilibrium, because strong consistency would force $\lambda^{high} = 1$. Thus, offering $(\overline{x}, 1 - \overline{x})$ is a better option for player 1 than just opting out, leading to a contradiction.) ■

Comment 1: The analysis of Proposition 6 is pretty similar to the one that would arise if player 1 could only propose a partition offer in $\{(\overline{x}, 1 - \overline{x}), (1, 0)\}$. Thus, analogy grouping here has the effect (through the working of the best-response correspondence) of reducing (discretizing) the offers considered by player 1 in equilibrium.

Comment 2: Another setup which would yield a similar equilibrium outcome is one in which the responder would not distinguish within the set of high offers (i.e., offers x such that $x \leq \overline{x}$) nor within the set of low offers (i.e., offers x such that $x > \overline{x}$).[32] However, a slight modification in which player 1 could no longer pick a deterministic offer, but could only affect the distribution of offers received by player 2 would highlight the difference between the two approaches.[33]

[32] See Dow (1991), Meyer (1991) and Rubinstein (1993) for investigations of coarse informational partitionings of this sort.

[33] For the sake of illustration, suppose that when player 1 picks x, player 2 receives the offer $(x, 1 - x)$ with a large probability, but also any offer $(y, 1 - y)$ with $y < x$ with a small probability. Consider first the analogy setting with the same partitioning as above, and assume that $1 - \overline{x} < v^{out}$. Player 1 will pick $\overline{x}$, and sometimes with small probability the deal will be accepted whenever the offer $(y, 1 - y)$ received by player 2 is such that $1 - y > v^{out}$. By contrast, in the setup where the responder has a coarse perception of offers (i.e., he cannot distinguish among offers $(y, 1 - y)$ such that $y \leq \overline{x}$, and similarly for offers $(y, 1 - y)$ such that $y > \overline{x}$), there is no agreement and player 2 always opts out. The point is that if player 1 were to pick $\overline{x}$, player 2 would reject any offer $(y, 1 - y)$, $y \leq \overline{x}$, because he would suspect (rightly in equilibrium) that such an offer is very likely to be $(\overline{x}, 1 - \overline{x})$ in which case he prefers opting out.

4 Discussion

4.1 Interpretation

The analogy partitioning used by player i should be thought of as standing for player i's information treatment at the learning stage.[34] More precisely, the analogy-based expectation equilibrium should be understood as the limiting outcome of a converging learning process in which each player i would only keep track of the average behavior of his opponent in every class α_i at every step of the learning process.

Note that the theory (and the above learning story) is silent about why player i only keeps track of that information (at the learning stage), as the analogy partitioning of player i is left exogenous. It may be so because during the learning stage somehow only this information is being received and (properly) stored by player i. Or it may be that due to past experiences, somehow player i thinks his analogy partition captures the essence of the determinants to his opponents' behavior. Or in a two player game, it may be that player i (wrongly) believes that the information sets of player j correspond to α_i and player i never thinks of checking whether his belief is supported by the behavior of player j.

The interpretation just given suggests some form of imperfection in the information treatment of the players. We wish to stress however that it is very different from imperfect recall as developed by Rubinstein (1991) and Piccione-Rubinstein (1997) (see also Dulleck-Oechssler 1997 for an application to the centipede game).[35]

To illustrate the difference, consider again the centipede game CP_K described in subsection 3.2. But assume that each player $i = 1, 2$ does *not* know at which node $N_i^{(k)}$, $k = 1, ...K$ he currently is (whereas players no longer form their expectations by analogy). For K large enough, an equilibrium in this setting is that each player i Passes with probability 1 in his unique memory/information set $I_i = \left\{ N_i^{(k)}, k = 1, ...K \right\}$.[36]

Imperfect recall explains in this case why players may Pass all the time in the centipede game. However, it does so by assuming that players do not perceive that there is an end (since players are assumed not to know at which node they currently are). In the analogy approach developed in subsection 3.2, players *do* know at which node they currently are. They Pass initially because they do not have an *accurate* expectation about when their opponent is to

[34] For another approach to imperfections in information treatment in a non-game-theoretic setup, see Mullainathan (2001).

[35] One can argue that games with imperfect recall fall in the class of games with incomplete information (see the discussion in Piccione-Rubinstein 1997).

[36] The point is that player i cannot adjust the best time for Taking, as he does not know in which $N_i^{(k)}$ he currently is. He prefers Passing always in this case.

Take (they only have an expectation about the average behavior of their opponent all over the game). Also, players do perceive that there is an end, as they consider Taking toward reaching the end of the game. Thus the two approaches have a very different interpretation, and only the analogy approach captures (in an endogenous way) the end effect in the finite horizon paradoxes.

4.2 Two principles on analogy partitioning

No structure was imposed so far on the analogy categorization. Understanding how individuals categorize contingencies to form their expectations is clearly beyond the scope of this paper (it is at the heart of a large body of the ongoing research of cognitive scientists, see Holyoak-Thagard 1995 and Dunbar 2000, for example). As a modest game-theoretic investigation, we now review two principles (for analogy partitioning) that may alternatively be viewed as attempts to refine the concept of analogy-based expectation equilibrium.

Analogy expectation and similar play:

An appealing idea seems to be that in order for player i to pool several nodes (j, h) into a single class of analogy, player i should himself consider playing in the same way in some pool of nodes. One difficulty is that in general player i need not move in the same nodes as player j, and therefore one should also worry about which nodes $h' \in H_i$ player i considers as being similar to nodes $h \in H_j$.

A class of situations in which this issue can be addressed simply is one in which whenever player i bundles two elements (j, h) and (j', h') into the same analogy class α_i, player i also has to move in h and h'. And the property is that player i behaves in the same way in nodes h and h'. An application of this idea is now being considered to illustrate the difference with the Subgame Perfect Nash Equilibrium concept:

Example 3: Consider the following two-stage two-player game. Player 1 moves first and chooses between the normal form game G or G'. In both G and G', players 1 and 2 move simultaneously, and in both G and G', player 1 chooses in $A_1 = \{U, D\}$, player 2 chooses in $A_2 = \{L, R\}$. We assume that U is a dominant strategy in both G and G' for player 1. Player 2's best-response to U is R in game G, whereas it is L in game G'. Finally, we assume that player 1 derives a higher payoff when (U, R) is played in game G than when (U, L) is played in game G'. And that player 1 derives a higher payoff when (U, L) is played in game G' than when (U, L) is played in game G.

The unique Subgame Perfect Nash Equilibrium is such that player 1 chooses game G and then (U, R) occurs.[37]

Suppose that player 1 puts in the same analogy class $(2, G)$ and $(2, G')$ in order to predict player 2's behavior. Note first that player 1 behaves in the same way in G and G' (he has the same dominant strategy in both games). Thus, the required property is satisfied. Second, it is readily verified that an equilibrium outcome in this analogy setting is that player 1 chooses G' (expecting player 2 to play L in both G and G'), since player 1 prefers (U, L) in game G' to (U, L) in game G. ■

All analogy classes must be reached:

Another property that may be of interest is that players structure their analogy classes so that each analogy class is reached with positive probability in equilibrium.[38] The next Proposition provides some insight about the effect of this principle in the centipede game CP_K considered in subsection 3.2.

Proposition 7 *Let* (σ, β) *be an analogy-based expectation equilibrium of* $(N, CP_K, \succsim_i, An)$ *where* $N = \{1, 2\}$ *denotes the set of players,* $\succsim_i$ *player i's preferences, and An the partitioning profile used by the players. Suppose that for all k,* $\frac{1}{2}a_{k-2} + \frac{1}{2}a_{k-1} > a_k$ *and* $\frac{1}{2}b_{k-2} + \frac{1}{2}b_{k-1} > b_k$*. If* σ *employs only pure strategies, and* **all** *analogy classes of both players are reached with positive probability according to* σ*, then the equilibrium outcome is that player* 1 **Takes** *in the last node* $N_1^{(1)}$*.*

5 Related Literature and Conclusion

This paper belongs to the tradition of incorporating elements of bounded rationality into game theory. Other approaches to bounded rationality (following the lead of Simon 1955) include the ε-equilibrium (Radner 1986), the quantal response equilibrium (McKelvey-Palfrey 1992-1995), limited foresight models (Jehiel 1995-1998-2001), games with procedurally rational players (Osborne-Rubinstein 1998), and more recently the (partially) cursed equilibrium (Eyster-Rabin 2000).[39]

Like this paper, models of limited foresight challenge the cognitive rationality of the players in that players do not know some aspects of the behavior of their opponents. But, the

[37] If he were to choose G' he would get the payoff attached to (U, L) in G', which is smaller.

[38] A possible psychological rationale for this is that players tend to prefer structuring analogy classes so that expectations can be checked on the equilibrium path (without trembling requirement).

[39] This review does not include approaches with fairness considerations (which mostly consist in postulating that preferences incorporate the well-being of other players).

implication of limited foresight is very different from that of analogy grouping.[40] In Osborne-Rubinstein (1998) too players do not rightly perceive the behavior of their opponent (players use a heuristic based on the idea that they play against Nature). But, unlike this paper, the interpretation is that players have an erroneous perception of the game being played (see also Greenberg 1996 and Camerer 1998 for an experimental account of misperception of games). Eyster-Rabin (2000)'s (partially) cursed equilibrium applies to static games with incomplete information and common values. It corresponds to a Nash-Bayes equilibrium of a modified static game in which preferences are either the original preferences with some exogenously given probability or modified preferences standing for the expected preferences over other players' types with the complementary probability.[41]

There are many facets to analogy thinking. Other approaches in economics include the axiomatic approaches of Rubinstein (1988) and Gilboa-Schmeidler (1994) about similarity and case-based decision theory, respectively (which derive representation theorems for some axiomatic).[42] These also include the automata theory developed for game theory by Rubinstein (1986), and Abreu-Rubinstein (1988) (see also Samuelson 2001 for a recent contribution with an explicit reference to the analogy interpretation).[43] It should be noted that none of these other approaches considers the treatment of expectations (as opposed to behaviors) by analogy.

This paper has proposed a solution concept for complicated interactions in which players use simplified representations of their opponents' behavior. And the resulting concept appears to be significantly different from previous solution concepts in game theory. Future research in this line should bear on how players group contingencies to form their analogy classes as well as on economic applications in which analogy thinking might play an important role (due to the complexity of the environment). Experimental investigations as to how players bundle contingencies at the learning stage are also required.[44]

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[40] For example, limited foresight cannot explain cooperation in the finitely repeated prisoner's dilemma.

[41] Eyster-Rabin (2000)'s model captures some aspects of a situation in which players would be imperfectly aware of the common value element of the game. However, it seems to me that the underlying learning required for their concept is as complex as in the standard setup, which makes it hard for a bounded rationality interpretation. Also, the approaches to partial sophistications are very different in Eyster-Rabin's and my approaches.

[42] The similarity function obtained in their representation is smoother than that corresponding to the analogy partitioning treatment adopted in this paper, but see Remark 3 at the end of Section 2.2.

[43] In the automaton setup, two different histories h and h' may induce the same state of player i's machine, and thus the same action of player i; Player i then acts in an analogous way in h and h'.

[44] A first step toward this end is to let the subjects play several times a given a game and consider various treatments regarding the feedback they receive after each round about others' behavior. Different feedback treatments should give rise to different long run outcomes according to the theory.

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Appendix

Proof of Proposition 3 (first part):

The two mentioned outcomes can be obtained as analogy-based expectation equilibrium outcomes.

(i) Observe first that the Subgame Perfect Nash Equilibrium outcome corresponds to the analogy-based expectation equilibrium (σ, β) in which for $i = 1, 2$, $\beta_i(\alpha_i) = Take$ and $\sigma_i(N_i^{(k)}) = Take$ for all $k = 1, ...K$.

(ii) Consider the strategy profile σ such that player 2 Passes always and player 1 Takes in the last period $2K$.

To be consistent with σ, the analogy-based expectation of player 1 must be that player 2 Passes with probability 1, i.e. $\beta_1(\alpha_1) = Pass$ (since player 2 Passes always when he has to move).

To be consistent with σ, the analogy-based expectation of player 2 must be that player 1 Passes with probability $\frac{K-1}{K}$, since (according to σ) each node $N_1^{(k)}$, $k = K, ...1$ is reached with probability 1, i.e. $P^{\sigma}(N_1^{(k)}) = 1$, (so that they have equal weighting), and player 1 Passes (with probability 1) in all nodes $N_1^{(k)}$, $k = K, ...2$ and Takes in node $N_1^{(1)}$. Thus, $\beta_2(\alpha_2) = \frac{K-1}{K} Pass + \frac{1}{K} Take$.

The (sequential) best-response of player 1 to the analogy-based expectation β_1 is to Take in the last node $N_1^{(1)}$. Thus, it is to play according to σ_1.

When condition (4) holds, the best-response of player 2 to the analogy-based expectation β_2 is to Pass always (since by Taking at $N_2^{(k+1)}$, player 2 would only get b_{2k+2}, which is less than the expected payoff he gets by Passing at $N_2^{(k+1)}$ and Taking at $N_2^{(k)}$, say, i.e. $\frac{K-1}{K}b_{2k} + \frac{1}{K}b_{2k+1} > b_{2k+2}$). Thus, it is to play according to σ_2.

Altogether the above considerations show that the assessment (σ, β) is an analogy-based expectation equilibrium. ■

Proof of Proposition 4:

(1) Suppose $\beta_i(\alpha_i) = \lambda^i.Pass + (1-\lambda^i).Take$ with $\lambda^i \geq \frac{1}{2}$ for $i = 1, 2$. Under condition (3),[45] it is readily verified that there exists an integer $\overline{m}$ such that for all $K > \overline{m}$, player i's best-response to β_i requires Passing (with probability 1) in the first $K - \overline{m}$ moves (at least) (because for some appropriately specified $\overline{m}$, Taking earlier is dominated by never Taking).

Suppose that players 1 and 2 Pass with probability 1 in the first node where they must move. The consistency condition implies that the analogy-based expectation of player i, $\beta_i(\alpha_i) = \lambda^i.Pass + (1-\lambda^i).Take$, should satisfy $\lambda^i \geq \frac{1}{2}$.

Together these two observations imply that the mapping

$$\beta = (\beta_1, \beta_2) \underset{\text{Best-response}}{\rightarrow} \sigma = (\sigma_1, \sigma_2) \underset{\text{Consistency}}{\rightarrow} (\beta_1 \langle \sigma \rangle, \beta_2 \langle \sigma \rangle)$$

has a fixed point such that $\lambda^i \geq \frac{1}{2}$ for $i = 1, 2$. Given the best-response to such analogy-based expectations, we may conclude.

(2) Suppose player i's strategy requires him to Pass with probability 1 in node $N_i^{(K)}$ for $i = 1, 2$. By the consistency requirement it should be that player i's analogy-based expectation $\beta_i(\alpha_i) = \lambda^i.Pass + (1-\lambda^i).Take$ satisfies $\lambda^i \geq \frac{1}{2}$ for $i = 1, 2$. The best-response to β_i is to Pass at least in the first $K - \overline{m}$ where he must move. ■

Proof of Proposition 5:

Given the assumed strategy profile σ and given that $\overline{m}$ is no larger than $\overline{g}$, for T large enough, the analogy-based expectations $\beta_i \langle \sigma \rangle$ that is consistent with σ should satisfy:

$$\begin{aligned} \beta_i \langle \sigma \rangle (\alpha_i) &\approx C \\ \beta_i \langle \sigma \rangle (\alpha_i') &= D \end{aligned}$$

It can be checked that the best-response to such a $\beta_i \langle \sigma \rangle$ is indeed σ_i.[46] ■

[45] Since all payoffs are integers satisfying (2), the differences $a_t - a_{t+2}$, $b_t - b_{t+2}$ are no smaller than 2.

[46] The sequence u_t has been constructed precisely for that purpose.

Proof of Proposition 7:

Take at node $N_1^{(1)}$ is a possible equilibrium outcome when players use the coarsest partition (see subsection 3.2). Since all classes of both players are then reached with positive probability, this outcome can be sustained in the way required by the Proposition.

Suppose that another outcome, i.e. player i Takes at node $N_i^{(k)}$ with $(i,k) \neq (1,1)$, were to emerge with the same requirements.

First, it cannot be that this outcome corresponds to the Subgame Perfect Nash Equilibrium outcome, since then no node $N_1^{(k)}$ would be reached, and thus at least one of the analogy classes of player 2 would not be reached in equilibrium.

If player i were to Pass at node $N_i^{(k)}$ this would lead to node $N_j^{(k')}$, $j \neq i$, with $k' = k$ if $i = 1$ and $k' = k - 1$ if $i = 2$. Since node $N_j^{(k')}$ is not reached in equilibrium and since all analogy classes must be reached with positive probability, it must be that there is an analogy class α_i of player i such that $(j, N_j^{(k')}) \in \alpha_i$ and $(j, N_j^{(k'')}) \in \alpha_i$ where $k'' < k'$ (nodes $N_j^{(k'')}$ with $k'' > k'$ are not reached).[47] Since at any node $N_j^{(k'')}$ with $k'' < k'$ player j Passes with probability 1 (remember that Take at node $N_i^{(k)}$ is the assumed outcome), it must be that the analogy-based expectation of player i satisfies

$$\beta_i(\alpha_i) = \lambda^i \cdot Pass + (1 - \lambda^i) \cdot Take \text{ with } \lambda^i \geq \frac{1}{2}.$$

But given this expectation (and given that for all k, $\frac{1}{2}a_{k-2} + \frac{1}{2}a_{k-1} > a_k$ and $\frac{1}{2}b_{k-2} + \frac{1}{2}b_{k-1} > b_k$), Taking at node $N_i^{(k)}$ cannot be a best-response to β_i (at node $N_i^{(k)}$, player i should strictly prefer Passing rather than Taking). This leads to a contradiction. ■

[47] There exists at least one such node because $(i,k) \neq (1,1)$.