Budget constraint

Consumers purchase goods $q$ from within a budget set $B$ of affordable bundles. In the standard model, prices $p$ are constant and total spending has to remain within budget $p'q \leq y$ where $y$ is total budget. Maximum affordable quantity of any commodity is $y/p_i$ and slope $\partial q_i / \partial q_j |_B = -p_j / p_i$ is constant and independent of total budget.

In practical applications budget constraints are frequently kinked or discontinuous as a consequence for example of taxation or non-linear pricing. If the price of a good rises with the quantity purchased (say because of taxation above a threshold) then the budget set is convex whereas if it falls (say because of a bulk buying discount) then the budget set is not convex.

Marshallian demands

The consumer’s chosen quantities written as a function of $y$ and $p$ are the Marshallian or uncompensated demands $q = f(y, p)$

Consider the effects of changes in $y$ and $p$ on demand for, say, the $i$th good:

- total budget $y$
  - the path traced out by demands as $y$ increases is called the income expansion path whereas the graph of $f_i(y, p)$ as a function of $y$ is called the Engel curve
  - we can summarise dependence in the total budget elasticity
    \[ \epsilon_i = \frac{y}{q_i} \frac{\partial q_i}{\partial y} = \frac{\partial \ln q_i}{\partial \ln y} \]
    - if demand for a good rises with total budget, $\epsilon_i > 0$, then we say it is a normal good and if it falls, $\epsilon_i < 0$, we say it is an inferior good
    - if budget share of a good, $w_i = p_i q_i / y$, rises with total budget, $\epsilon_i > 1$, then we say it is a luxury or income elastic and if it falls, $\epsilon_i < 1$, we say it is a necessity or income inelastic

- own price $p_i$
  - the path traced out by demands as $p_i$ increases is called the offer curve whereas the graph of $f_i(y, p)$ as a function of $p_i$ is called the demand curve
- we can summarise dependence in the (uncompensated) own price elasticity

\[ \eta_{ii} = \frac{p_i}{q_i} \frac{\partial q_i}{\partial p_i} = \frac{\partial \ln q_i}{\partial \ln p_i} \]

- if uncompensated demand for a good rises with own price, \( \eta_{ii} > 0 \), then we say it is a Giffen good
- if budget share of a good rises with price, \( \eta_{ii} > -1 \), then we say it is price inelastic and if it falls, \( \eta_{ii} < -1 \), we say it is price elastic

• other price \( p_j, j \neq i \)

- we can summarise dependence in the (uncompensated) cross price elasticity

\[ \eta_{ij} = \frac{p_j}{q_i} \frac{\partial q_i}{\partial p_j} = \frac{\partial \ln q_i}{\partial \ln p_j} \]

- if uncompensated demand for a good rises with the price of another, \( \eta_{ij} > 0 \), then we can say it is an (uncompensated) substitute whereas if it falls with the price of another, \( \eta_{ij} < 0 \), then we can say it is an (uncompensated) complement. These are not the best definitions of complementarity and substitutability however since they may not be symmetric ie \( q_i \) could be a substitute for \( q_j \) while \( q_j \) is a complement for \( q_i \). A better definition, guaranteed to be symmetric, is one based on the concept of compensated demands to be introduced below.

**Adding up**

We know that demands must lie within the budget set:

\[ p' f(y, p) \leq y. \]

If consumer spending exhausts the total budget then this holds as an equality,

\[ p' f(y, p) = y, \]

which is known as adding up.

By adding up

• not all goods can be inferior
• not all goods can be luxuries
• not all goods can be necessities

Also certain specifications are ruled out for demand systems. It is not possible, for example, for all goods to have constant income elasticities unless these elasticities are all 1. Otherwise \( p_i q_i = A_i y^{\alpha_i} \), say, and 1 = \( \sum_i A_i \alpha_i y^{\alpha_i - 1} \) for all budgets \( y \) which is impossible unless all \( \alpha_i = 1 \). This does not rule out constant elasticities for individual goods.
There are also restrictions on price effects - for example, if price of some good goes up then purchases of some good must be reduced so no good can be a Giffen good unless it has strong complements.

**Homogeneity**

Multiplying $y$ and $p$ by the same factor does not affect preferences or the budget constraint so choices should not be affected either, assuming that $y$ and $p$ influence choice only through the budget constraint. Marshallian demands should therefore be *homogeneous* of degree zero:

$$f(\lambda y, \lambda p) = f(y, p)$$

for any $\lambda > 0$.

**Preferences**

Suppose the consumer has a preference relation $\succeq$ where $q^A \succeq q^B$ means “$q^A$ is at least as good as $q^B$”. For the purpose of modelling demand this can be construed as an inclination to choose the bundle $q^A$ over the bundle $q^B$. For modelling welfare effects the interpretation needs to be strengthened to include a link to consumer wellbeing.

A weak preference relation suffices to define strict preference $\succ$ and indifference $\sim$ if we let

- $\succeq$ and $\preceq$ be equivalent to $\sim$,
- $\succsim$ and $\sim$ be equivalent to $\succ$.

We want the preference relation to provide a basis to consistently identify a set of most preferred elements in any possible budget set and for this we need assumptions.

- **Completeness** Either $q^A \succsim q^B$ or $q^B \succsim q^A$. This ensures that choice is possible in any budget set.
- **Transitivity** $q^A \succsim q^B$ and $q^B \succsim q^C$ implies $q^A \succsim q^C$. This ensures that there are no cycles in preferences within any budget set.

Together these ensure that the preference relation is a *preference ordering*.

**Indifference curves**

For any bundle $q^A$ define

- the weakly preferred set $R(q^A)$ as all bundles $q^B$ such that $q^B \succeq q^A$
- the indifferent set $I(q^A)$ as all bundles $q^B$ such that $q^B \sim q^A$
To make these sets well behaved we make the technical assumption:

- **Continuity** If \( q^A \succeq q^B \) and \( q^B \succeq q^C \) then there is a bundle indifferent to \( q^B \) on any path joining \( q^A \) to \( q^C \). This rules out discontinuous jumps in preferences.

Continuity is violated by the example of *lexicographic* preferences. Say that there are two goods \( q_1 \) and \( q_2 \) and that the consumer prefers one bundle to another if and only if it either has more of \( q_1 \) or the same amount of \( q_1 \) and more of \( q_2 \). Such preferences do not satisfy continuity and indifferent sets are single points.

A further assumption rules out consumers ever being fully satisfied:

- **Nonsatiation** Given any bundle there is always some direction in which changing the bundle will make the consumer better off.

If this is true then indifferent sets have no “thick” regions to them and we can visualise them as indifference curves.

### Utility functions

A utility function \( u(q) \) is a representation of preferences such that \( q^A \succeq q^B \) if and only if \( u(q_A) \geq u(q_B) \). A utility function exists if preferences give a continuous ordering.

Utility functions are not however unique since if \( u(q) \) represents certain preferences then any increasing transformation \( \phi(u(q)) \) also represents the same preferences. We say that utility functions are *ordinal*.

### Shape of indifference curves

We now consider assumptions which put some actual shape on indifference curves. For example:

- **Monotonicity** Larger bundles are preferred to smaller bundles.

Given monotonicity, indifference curves must slope down. This slope is known as the marginal rate of substitution (MRS).

Monotonicity corresponds to increasingness of the utility function \( u(q) \). An indifference curve is defined by \( u(q) \) being constant and therefore the MRS is given by

\[
\text{MRS} = \frac{dq_2}{dq_1} \bigg|_{u} = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2}
\]

which is obviously negative if \( \partial u/\partial q_1, \partial u/\partial q_2 > 0 \).

- **Convexity** \( \lambda q^A + (1 - \lambda)q^B \succeq q^B \) if \( q^A \succeq q^B \) and \( 1 \geq \lambda \geq 0 \). This says that weakly preferred sets are convex or, equivalently, MRS is diminishing.
Convexity can be interpreted as capturing taste for variety. It says that a consumer will always prefer to mix any two bundles between which they are indifferent. The corresponding property of the utility function is known as quasiconcavity.

**Homotheticity**

Preferences are said to be homothetic if $q^A \sim q^B$ implies that $\lambda q^A \sim \lambda q^B$ for any $\lambda > 0$. Graphically this means that higher indifference curves are magnified versions of lower ones from the origin. This is a strong restriction that would rarely be made in practice but it is useful to consider as a reference case. It is not a restriction on the shape of any one indifference curve but on the relationship between indifference curves within an indifference map.

If preferences are homothetic then marginal rates of substitution are constant along rays through the origin. This is only true for homothetic preferences and this is usually an easy way to check whether given preferences are homothetic.

If there exists a homogeneous utility representation $u(q)$ where $u(\lambda q) = \lambda u(q)$ then preferences can be seen to be homothetic. Since increasing transformations preserve the properties of preferences, then any utility function which is an increasing function of a homogeneous utility function also represents homothetic preferences.

**Quasilinearity**

Quasilinearity is another strong restriction based on a similar idea. Preferences are quasilinear if

$\left( \begin{array}{c} q_1^A \\ q_2^A \\ \vdots \\ q_n^A \\
\end{array} \right) \sim \left( \begin{array}{c} q_1^B \\ q_2^B \\ \vdots \\ q_n^B \\
\end{array} \right)$ implies $\left( \begin{array}{c} q_1^A + \lambda \\ q_2^A + \lambda \\ \vdots \\ q_n^A + \lambda \\
\end{array} \right) \sim \left( \begin{array}{c} q_1^B + \lambda \\ q_2^B + \lambda \\ \vdots \\ q_n^B + \lambda \\
\end{array} \right)$.

In other words adding the same amount to one particular good preserves indifference. This means that higher indifference curves are parallel translations of lower ones. In this case, marginal rates of substitution are constant along lines parallel to axes.

If there exists a utility representation $u(q)$ such that $u(q_1, q_2, \ldots) = q_1 + F(q_2, q_3, \ldots)$, say, then preferences are quasilinear. This is also true of any utility functions which are increasing transformations of functions with this property.

**Some Examples**

- **Perfect substitutes** $u(q_1, q_2) = aq_1 + bq_2$: The MRS is $-a/b$ and is constant. Indifference curves are parallel straight lines. These are the only preferences which are homothetic and quasilinear.

- **Perfect complements** $u(q_1, q_2) = \min[q_1, q_2]$: Indifference curves are L-shaped with the kinks lying on a ray through the origin of slope $a/b$. These preferences are homothetic but not quasilinear.
• **Cobb-Douglas**: \( u(q_1, q_2) = a \ln q_1 + b \ln q_2 \): Preferences are homothetic, indifference curves are smooth and MRS \( aq_2/bq_1 \) is diminishing

### Revealed preference

Suppose the consumer chooses \( q^A \) at prices \( p^A \) when \( q^B \) was cheaper:

\[
p'^A q^A > p'^A q^B.
\]

We say that \( q^A \) is (directly) revealed preferred to \( q^B \). The **Weak Axiom of Revealed Preference (WARP)** says that the consumer would never then choose \( q^B \) at prices \( p^B \) when \( q^A \) was affordable:

\[
p'^B q^A \leq p'^B q^B.
\]

This is an implication of consumer optimisation.

The **Strong Axiom of Revealed Preference (SARP)** says that there should be no cycles in revealed preference eg we should never find \( q^A \) revealed preferred to \( q^B \), \( q^B \) revealed preferred to \( q^C \) and \( q^C \) revealed preferred to \( q^A \) (or any longer cycle). This is equivalent to consumer optimisation.

### Negativity

Suppose that as prices change from \( p^A \) to \( p^B \) the consumer is compensated in the sense that their total budget is adjusted to maintain affordability of the original bundle, this is known as **Slutsky compensation**. Then choices before and after satisfy \( p'^B q^A = p'^B q^B \). But the later choice cannot then have been cheaper at the initial prices or the change would violate WARP. Hence \( p'^A q^A \leq p'^A q^B \).

By subtraction we get **negativity**:

\[
(p^B - p^A)(q^B - q^A) \leq 0.
\]

This shows a sense in which price changes and quantity changes must move, on average, in opposite directions if the consumer is compensated.

If we consider the case where the price of only one good changes then we see that this implies that Slutsky compensated effects of own price rises must be negative. In other words, Slutsky compensated demand curves necessarily slope down.

Note that convexity was not assumed anywhere in the argument.

### Slutsky equation

The Slutsky compensated demand function given initial bundle \( q^A \) is defined by

\[
g(q^A, p) = f(p' q^A, p)
\]
that is to say it is the amount demand if budget is constantly adjusted to keep $q^A$ affordable. Differentiating establishes a relationship between Slutsky compensated and Marshallian price effects

$$\frac{\partial g_i}{\partial p_j} = q^A_j \frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial p_j}.$$  

The difference is the income effect $q^A_j \frac{\partial f_i}{\partial y}$ and it is positive if the good is normal. Hence, since the Slutsky compensated effect has been shown to be negative, so must be the Marshallian effect for normal goods. This is the Law of Demand.

The Slutsky equation is highly useful. Its importance is that it allows testing of restrictions regarding compensated demands since it shows how to calculate compensated effects from the sort of uncompensated effects estimated in applied demand analysis.

**Tangency condition**

If all goods are chosen in positive quantities and preferences are convex then the solution to the consumer’s optimisation problem is at a tangency between an indifference curve and the boundary of the budget set.

This is true even for non-linear budget sets. However if the budget set is linear (or indeed simply convex) then we know that such a tangency is unique so finding one guarantees that we have found the best choice for the consumer. If the budget set is not convex, on the other hand, then there can be multiple tangencies and the optimum can typically be found only by comparing the level of utility at each of them.

The nature of Marshallian demands can then be inferred by moving the budget constraint to capture changes in $y$ and $p$ and tracing out movement of the tangency.

**Demand under homotheticity and quasilinearity**

As income increases, slopes of budget constraints do not change. Income expenditure paths traced out by the tangencies as incomes are increased therefore all occur at points with the same MRS.

Homotheticity and quasilinearity are each characterised by the nature of paths along which MRS is constant and therefore each give rise to income expansion paths of particular shapes.

- **Homotheticity** MRS is constant along rays through the origin so income expansion paths are rays through the origin. Quantities consumed are proportional to total budget $y$ given prices and budget shares are independent of $y$.

- **Quasilinearity** MRS is constant along lines parallel to one of the axes so income expansion paths are parallel to one of the axes. Quantities demanded of all but one of the goods are independent of $y$ (for interior solutions).
Constrained optimisation

Mathematically, Marshallian demands solve

\[
\max_q u(q) \quad \text{s.t.} \quad p'q \leq y
\]

This can be solved by finding stationary points of the Lagrangean

\[
u(q) - \lambda(p'q - y).
\]

For interior solutions, first order conditions require

\[
\frac{\partial u}{\partial q_i} = \lambda p_i
\]

which imply

\[
MRS = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.
\]

This is a confirmation of the tangency condition - the slope of indifference curve and budget constraint are equal at interior solutions.

Duality

In comparison with the primal problem

\[
\max_q u(q) \quad \text{s.t.} \quad p'q \leq y
\]

consider now the dual problem of minimising the expenditure necessary to reach a given utility

\[
\min_q p'q \quad \text{s.t.} \quad u(q) \geq v
\]

Solutions to this problem are Hicksian or compensated demands \( q = g(v, p) \).

The problem can be solved by finding stationary points of the Lagrangean

\[
p'q - \mu(u(q) - v).
\]

which, for interior solutions, gives first order conditions requiring

\[
p_i = \mu \frac{\partial u}{\partial q_i}
\]

and therefore

\[
MRS = -\frac{\partial u/\partial q_i}{\partial u/\partial q_j} = -\frac{p_i}{p_j}.
\]

Note that this is exactly the same tangency condition encountered in solving the primal problem.

We can define important functions giving the values of the primal and dual problems. The value of the maximised utility function as a function of \( y \) and
$p$ can be found by substituting the Marshallian demands back into the direct utility function $u(q)$. We call this the indirect utility function

$$v(y, p) = u(f(y, p)) = \max_q u(q) \text{ s.t. } p'q \leq y$$

The value of the minimised cost in the dual problem as a function of $\nu$ and $p$ can be found by costing the Hicksian demands. We call this the expenditure function or cost function

$$c(\nu, p) = p'g(\nu, p) = \min_q p'q \text{ s.t. } u(q) \geq \nu.$$

The duality between the two problems can be expressed by noting the equality of the quantities solving the two problems

$$f(c(\nu, p), p) = g(\nu, p) \quad f(y, p) = g(v(y, p), p)$$

or noting that $v(y, p)$ and $c(\nu, p)$ are inverses of each other in their first arguments

$$v(c(\nu, p), p) = \nu \quad c(v(y, p), p) = y.$$

**Expenditure function**

The expenditure function $c(\nu, p)$ has the properties that

- it is increasing in each price in $p$ and in $\nu$
- it is homogeneous of degree one in prices $p$, $c(\nu, \lambda p) = \lambda c(\nu, p)$. The Hicksian demands are homogeneous of degree zero so the total cost of purchasing them must be homogeneous of degree one

$$c(\nu, \lambda p) = \lambda p'g(v, \lambda p) = \lambda p'g(v, p) = \lambda c(\nu, p)$$

**Indirect utility function**

The properties of the indirect utility function $v(y, p)$ correspond exactly to those of the expenditure function given that the two are inverses of each other. In particular it is

- increasing in $y$ and decreasing in each element of $p$
- homogeneous of degree zero in $y$ and $p$:

$$v(\lambda y, \lambda p) = v(y, p)$$

This should be apparent also from the homogeneity properties of Marshallian demands.

**Shephard’s Lemma**
Among the most useful features of these functions are their simple links to the associated demands. For example, it is possible to get from the expenditure function to the Hicksian demands simply by differentiating.

Since \( c(v, p) = p' g(v, p) \)

\[
\frac{\partial c(v, p)}{\partial p_i} = g_i(v, p) + p' \frac{\partial g(v, p)}{\partial p_i} = g_i(v, p) + \mu \sum_i \frac{\partial u}{\partial q_i} \frac{\partial g(v, p)}{\partial p_i} = g_i(v, p)
\]

using the first order condition for solving the cost minimisation problem and the fact that utility is held constant in that problem.

This is known as Shephard’s Lemma. Its importance is that it allows compensated demands to be deduced simply from the expenditure function.

**Roy’s Identity**

Since \( v(c(v, p), p) = v \)

\[
\frac{\partial v(y, p)}{\partial p_i} + \frac{\partial v(y, p)}{\partial y} \frac{\partial c(u, p)}{\partial p_i} = 0
\]

\[
\Rightarrow - \frac{\partial v(y, p)}{\partial p_i} = g_i(v(y, p), p)
\]

\[
= f_i(y, p)
\]

using Shephard’s Lemma.

This is Roy’s identity and shows that uncompensated demands can be deduced simply from the indirect utility function by differentiation.

In many ways it is easier to derive a system of demands by beginning with well specified indirect utility function \( v(y, p) \) or expenditure function \( c(v, p) \) and differentiating than by solving a consumer problem directly given a direct utility function \( u(q) \).

**Slutsky equation, again**

Since \( g(v, p) = f(c(v, p), p) \)

\[
\frac{\partial g_i(v, p)}{\partial p_j} = \frac{\partial f_i(y, p)}{\partial p_j} + \frac{\partial f_i(y, p)}{\partial y} \frac{\partial c(v, p)}{\partial p_j}
\]

\[
= \frac{\partial f_i(y, p)}{\partial p_j} + \frac{\partial f_i(y, p)}{\partial y} f_j(y, p)
\]

Notice that this is the same as the Slutsky equation derived earlier for Slutsky-compensated demands. Hicks-compensated price derivatives are the
same as Slutsky-compensated price derivatives since the two notions of compensation coincide at the margin.

This means the results derived earlier can simply be carried over. Hicksian demands therefore also satisfy negativity at the margin. In particular

\[
\frac{\partial g_i(v, p)}{\partial p_i} \leq 0.
\]

(Negativity can actually be stated slightly more strongly than this, involving also restrictions on cross-price effects, but this is the most important implication).

**Slutsky symmetry**

There is one more property of Hicksian demands that can now be deduced. From Shephard’s Lemma

\[
\frac{\partial g_i(v, p)}{\partial p_j} = \frac{\partial^2 c(v, p)}{\partial p_i \partial p_j} = \frac{\partial g_j(v, p)}{\partial p_i}
\]

Compensated cross-price derivatives are therefore also symmetric. Holding utility constant, the effect of increasing the price of one good on the quantity chosen of another is numerically identical to the effect of increasing the price of the other good on the quantity chosen of the first good.

This shows that notions of complementarity and substitutability are consistent between demand equations if using compensated demands and provides a strong argument for defining complementarity and substitutability in such terms. This would not be true if using uncompensated demands because income effects are not symmetric.

**Demand restrictions**

To summarise, if demands are consistent with utility maximising behaviour then they have the following properties

- Adding up: Demands must lie on the budget constraint and therefore

\[
p' f(y, p) = y
\]

\[
p' g(v, p) = c(v, p)
\]

- Homogeneity: Increasing all incomes and prices in proportion leaves the budget constraint and therefore demands unaffected

\[
f_i(y, p) = f_i(\lambda y, \lambda p)
\]

\[
g_i(v, p) = g_i(v, \lambda p)
\]

- Negativity of compensated own price effects: In particular, a compensated increase in any good’s price can only reduce demand for that good

\[
\frac{\partial q_i}{\partial p_i} = \frac{\partial f_i}{\partial p_i} + f_i \frac{\partial f_i}{\partial y} \leq 0
\]
• Symmetry of compensated cross price effects:

\[ \frac{\partial g_i}{\partial p_j} = \frac{\partial g_j}{\partial p_i} \]

If demands satisfy these restrictions then there is a utility function \( u(q) \) which they maximise subject to the budget constraint. We say demands are integrable. These are all the restrictions required by consumer optimisation.

We know a system of demands is integrable if any of the following hold:

• They were derived as solutions to the dual or primal problem given a well specified direct utility function

• They were derived by Shephard’s Lemma from a well specified cost function or they were derived by Roy’s identity from a well specified indirect utility function

• They satisfy adding up, homogeneity, symmetry and negativity.

The connections between the concepts discussed can be summarised in the diagram below:

Utility maximisation problem
\[
\max_q u(q) \quad s.t. \quad p'q \leq y
\]
Uncompensated demands
\[
q_i = f_i(y, p) \quad i = 1, 2, \ldots
\]
Indirect utility function
\[
v(y, p)
\]
Some worked examples of consumer choice problems

• Example 1:
The direct utility function is

\[ u(q_1, q_2) = (q_1 - a_1)(q_2 - a_2) \]

The tangency condition defining optimum consumer choice is

\[
MRS = -\frac{\partial u/\partial q_1}{\partial u/\partial q_2} = -\frac{q_2 - a_2}{q_1 - a_1} = -\frac{p_1}{p_2}
\]

\[ \Rightarrow p_1q_1 - p_1a_1 = p_2q_2 - p_2a_2 \]
Substituting into the budget constraint
\[ y = 2p_1q_1 - p_1a_1 + p_2a_2 \]
and thus uncompensated demands are
\[
\begin{align*}
    f_1(y, p_1, p_2) &= \frac{(y + p_1a_1 - p_2a_2)}{2p_1} = a_1 + \frac{(y - p_1a_1 - p_2a_2)}{2p_1} \\
    f_2(y, p_1, p_2) &= \frac{(y - p_1a_1 + p_2a_2)}{2p_2} = a_2 + \frac{(y - p_1a_1 - p_2a_2)}{2p_2}
\end{align*}
\]

Substituting into the direct utility function gives the indirect utility function
\[
v(y, p_1, p_2) = \frac{(f_1(y, p_1, p_2) - a_1)(f_2(y, p_1, p_2) - a_2)}{2p_1} = \frac{(y - p_1a_1 - p_2a_2)}{2p_1}
\]

Inverting \(v(y, p_1, p_2)\) in \(y\) then gives the expenditure function
\[
c(v, p_1, p_2) = 2\sqrt{vp_1p_2} + p_1a_1 + p_2a_2
\]
The compensated demands are then most easily found by differentiating \(c(v, p_1, p_2)\) (using Shephard’s Lemma) or by substituting \(c(v, p_1, p_2)\) into the uncompensated demands
\[
\begin{align*}
    g_1(v, p_1, p_2) &= \frac{\partial c(v, p_1, p_2)}{\partial p_1} = f_1(c(v, p_1, p_2), p_1, p_2) = a_1 + \frac{\sqrt{v}p_2}{p_1} \\
    g_2(v, p_1, p_2) &= \frac{\partial c(v, p_1, p_2)}{\partial p_2} = f_2(c(v, p_1, p_2), p_1, p_2) = a_2 + \frac{\sqrt{v}p_1}{p_2}
\end{align*}
\]

- **Example 2:**
  Direct utility is
  \[ u(q_1, q_2) = \ln q_1 + q_2 \]
  Preferences are quasilinear.
  The tangency condition is
  \[
  MRS = -\frac{\partial u}{\partial q_1} = -\frac{1}{q_2} = -\frac{p_1}{p_2} \\
  \Rightarrow p_1q_1 = p_2
  \]
  This defines an interior optimum assuming \(y > p_2\).
  Hence, directly and by substituting into the budget constraint, uncompensated demands are
  \[
  \begin{align*}
    f_1(y, p_1, p_2) &= \frac{p_2}{p_1} \\
    f_2(y, p_1, p_2) &= \frac{(y/p_2) - 1}{p_2}
  \end{align*}
  \]
The uncompensated demand for the first good is independent of total budget $y$.

Substituting into the direct utility function gives the indirect utility function

$$v(y, p_1, p_2) = \ln f_1(y, p_1, p_2) + f_2(y, p_1, p_2)$$

$$= \ln(p_2/p_1) + (y/p_2) - 1$$

Inverting in $y$ gives the expenditure function

$$c(v, p_1, p_2) = p_2(v - \ln(p_2/p_1) + 1)$$

Differentiating or substituting then gives the compensated demands

$$g_1(v, p_1, p_2) = \frac{\partial c(v, p_1, p_2)}{\partial p_1} = f_1(c(v, p_1, p_2), p_1, p_2) = \frac{y}{p_1 + p_2}$$

$$g_2(v, p_1, p_2) = \frac{\partial c(v, p_1, p_2)}{\partial p_2} = f_2(c(v, p_1, p_2), p_1, p_2) = v - \ln(p_2/p_1)$$

The compensated demand for the first good is independent of $v$.

- **Example 3:**

Direct utility

$$u(q_1, q_2) = \min[q_1, q_2]$$

Goods are perfect complements and at the optimum

$$q_1 = q_2$$

$$\Rightarrow f_1(y, p_1, p_2) = f_2(y, p_1, p_2) = \frac{y}{p_1 + p_2}$$

Substituting into the direct utility function gives the indirect utility function

$$v(y, p_1, p_2) = \min[f_1(y, p_1, p_2), f_2(y, p_1, p_2)]$$

$$= \frac{y}{p_1 + p_2}$$

Inverting in $y$ gives the expenditure function

$$e(v, p_1, p_2) = v(p_1 + p_2)$$

Differentiating or substituting gives the compensated demands

$$g_1(v, p_1, p_2) = \frac{\partial c(v, p_1, p_2)}{\partial p_1} = v = \frac{\partial c(v, p_1, p_2)}{\partial p_2} = g_2(v, p_1, p_2)$$
Consumer surplus

By Roy’s identity the effect of a small change in price \( p_i \) on utility is proportional to the quantity consumed

\[
\frac{\partial v}{\partial p_i} = \frac{\partial v}{\partial y} q_i.
\]

The horizontal distance of the demand curve from the vertical axis is therefore an indicator of the marginal welfare cost of increasing price. If \( \partial v / \partial y \) is constant then the effect of increasing the price to the point where none of the good is demanded is therefore a triangular area underneath a demand curve. We call this consumer surplus.

What sort of demand curve do we need to use to keep \( \partial v / \partial y \) constant? If preferences are quasilinear then this will be true of the Hicksian demand curve. Even if preferences are not quasilinear the general idea behind calculating consumer surplus as the area under the compensated demand curve usually still gives a reasonable approximation to a good measure of welfare.

Cost of living indices

The expenditure function is the ideal concept for comparing cost of living. We can define a cost of living index as the ratio of the minimum cost of reaching a given utility in two periods. Say that we are comparing current prices \( p^A \) with prices in a base period \( p^B \). Then the cost of living index is the ratio of expenditure functions

\[
T(v, p^A, p^B) = \frac{c(v, p^A)}{c(v, p^B)}
\]

Notice that such a cost of living index depends on the utility level \( v \) at which we make the comparison. Must this be so? There are only two case in which not:

- If prices are proportional \( p^A = \lambda p^B \) then the cost of living index is equal to \( \lambda \) at all \( v \), whatever preferences,
- If preferences are homothetic then the cost of living index is equal at all \( v \), whatever prices.

Two common approximations to the true index are used. Both compare the cost of purchasing a fixed bundle of goods.

The Laspeyres index is the ratio of the costs of purchasing the base period bundle \( q^B \)

\[
L(p^A, p^B) = \frac{p^A' q^B}{p^B' q^B}.
\]

If \( L(p^A, p^B) < 1 \) and consumer’s total expenditure is unchanged then the consumer can afford the base bundle and cannot be worse off.
We also know that \( p^B q^B = c(v^B, p^B) \) where \( v^B \) is the base period utility and also that \( p^A q^B \) cannot be less than the minimum cost of attaining \( u^B \) in the current period (since \( q^B \) gives utility \( v^B \) but not necessary most cheaply at current prices). Hence the Laspeyres index is greater than the true cost of living index at base period utility,

\[
L(p^A, p^B) \geq T(v^B, p^A, p^B).
\]

This is so because consumers are free to substitute away from goods which become more expensive and therefore evaluating the cost at a fixed bundle exaggerates the impact on cost of living.

The Paasche index is the ratio of the costs of purchasing the current period bundle \( q^A \)

\[
P(p^A, p^B) = \frac{p^A q^A}{p^B q^A}.
\]

By similar arguments the consumer must be worse off if \( P(p^A, p^B) > 1 \) and total expenditure is unchanged. Likewise the Paasche index can be shown to be less than the true cost of living index at current utility \( v^A \),

\[
P(p^A, p^B) \leq T(v^A, p^A, p^B).
\]

**Buying and selling**

Suppose an individual has endowments \( \omega = (\omega_1, \omega_2, ...) \) of goods. The consumer problem becomes

\[
\max_q u(q) \quad \text{s.t.} \quad p' q \leq y + p' \omega
\]

Demands are now

\[
q_i = f_i(y + p' \omega, p) \quad i = 1, 2, \ldots
\]

where \( f_i(\cdot) \) is the standard uncompensated demand function.

Changes in endowments have effects like income effects. Changes in prices have the usual effects plus an effect due to the change in the value of the individual’s endowment - the endowment income effect. Specifically

\[
\frac{\partial q_i}{\partial p_i} = \frac{\partial f_i}{\partial p_i} + \frac{\partial f_i}{\partial y} \omega_i = \frac{\partial g_i}{\partial p_i} - (q_i - \omega_i) \frac{\partial f_i}{\partial y}
\]

where \( g_i(v, p) \) is the usual compensated demand function. This extends the Slutsky equation to the case of demand with endowments. Notice that the sign of the income effect depends upon whether the individual is a net buyer \( (q_i > \omega_i) \) as in the usual case or a net seller \( (q_i < \omega_i) \). An increase in the price
of a normal good can now increase demand if the individual is a net seller and the endowment income effect is therefore strong enough.

Note that there is an important revealed preference argument establishing that a seller will never become a buyer if the price rises and a buyer will never become a seller if the price fails. In each case, such a change is not possible since it would involve consuming a bundle available before the change when the bundle then chosen remains affordable.

### Labour supply

The prime example of the importance of considering demand with endowments is the analysis of labour supply. Suppose an individual has preferences over hours not working (“leisure”) \( h \) and consumption \( c \). He has unearned income of \( m \) and endowment of time \( T \). The price of consumption is \( p \) and the nominal wage is \( w \). The individual’s budget constraint is

\[
pc + wh = m + wT
\]

which may appear more familiar if written in terms of hours worked \( l = T - h \):

\[
pc = m + wl.
\]

The value of endowments in this context \( m + wT \) is referred to as full income.

Demand for leisure can be written as an uncompensated demand function, dependent on full income, wage and output price

\[
h = f(m + wT, w, p)
\]

or as a compensated demand function

\[
h = g(v, w, p).
\]

The Slutsky equation for leisure is

\[
\frac{\partial h}{\partial w} \bigg|_m = \frac{\partial g}{\partial w} - (h - T) \frac{\partial h}{\partial m}.
\]

Since the individual sells time \( (h < T) \) the income effect of a wage change is opposed to the compensated effect if leisure is normal.

Rephrasing in the more familiar terms of labour supply \( l \)

\[
\frac{\partial l}{\partial w} \bigg|_m = \frac{\partial l}{\partial w} \bigg|_v + l \frac{\partial l}{\partial m}.
\]

### Intertemporal choice

Another example of demand with endowments is analysis of intertemporal choice. Suppose an individual has preferences over consumption when young...
c_0 and consumption when old c_1. He has endowed income of y_0 and y_1 in the two periods. (If necessary, bequests received can be treated as part of y_0 and bequests given as part of c_1). Assume no uncertainty about the future. If the real interest rate on bonds linking the two periods is equal to r for both lending and borrowing then the budget constraint is
\[ c_0 + \frac{c_1}{1 + r} = y_0 + \frac{y_1}{1 + r}. \]
which implies that the present value of consumption must equal the present value of income.

Demand for current consumption is
\[ c_0 = f_0(y_0 + \frac{y_1}{1 + r}, r) \]

The effect of interest rate changes clearly depend upon whether the individual is a saver or a borrower since this determines the sign of the income effect. Note that an interest rate rise will never induce a saver to become a borrower and an interest rate fall will never induce a borrower to become a saver.

Often it is assumed that the utility function can be written as the sum of utility contributions from the different periods with similar within-period utility functions but with future utility discounted. Thus
\[ u(c_0, c_1) = \nu(c_0) + \frac{1}{1 + \delta} \nu(c_1) \]
where \( \nu(.) \) is the within-period utility function and \( \delta \) is a subjective discount rate. Convexity of preferences, which amounts here to a desire to smooth consumption over the life-cycle, requires \( \nu(.) \) to be concave ie \( \nu''(.) < 0 \).

Maximising such a utility function subject to the lifetime budget constraint
\[ \max_{c_0} \nu(c_0) + \frac{1}{1 + \delta} \nu(y_1 + (y_0 - c_0)(1 + r)) \]
gives first order condition
\[ \frac{\nu'(c_0)}{\nu'(c_1)} = \frac{1 + r}{1 + \delta}. \]
Given concavity of \( \nu(.) \), if \( r = \delta \), so that subjective discounting matches the market interest rate and impatience cancels out the market incentive to save, then \( c_0 = c_1 \) and the consumption stream is flat. If \( r > \delta \) then \( c_0 < c_1 \) and if \( r < \delta \) then \( c_0 > c_1 \).

**Asset choice**

Suppose that as well as investing in bonds with fixed return of r the individual can also invest in another asset - say a family enterprise. If X is placed in the family enterprise in the first period then suppose \( F(X) \) is returned in the second period.
The optimisation problem now has two dimensions
\[ \max_{c_0,X} \nu(c_0) + \frac{1}{1 + \delta} \nu(y_1 + F(X) + (y_0 - c_0 - X)(1 + r)) \]
and first order conditions (assuming an interior solution)

\[
\frac{\nu'(c_0)}{\nu'(c_1)} = \frac{1 + r}{1 + \delta} \\
F'(X) = (1 + r)
\]

Note that the solution to the financial decision is independent of intertemporal preferences. The individual invests in the family enterprise until the marginal rate of return falls to the market interest rate. This maximises the present value of the individual’s asset portfolio and the first order condition for optimum consumption choice given that present value is as in the simpler problem above.

The simplicity of the investment decision is a consequence of assuming away issues concerning risk, liquidity and so on.

**Uncertainty**

Extending the standard analysis to the case of uncertainty involves regarding quantities consumed in different uncertain states of the world as different goods. Preferences will depend on perceived probabilities of states of the world occurring. Budget constraints depend on the mechanisms available for managing risk.

Some examples of budget constraints in circumstances involving risk are:

- An individual with an asset worth \( A \) faces a probability \( \pi \) of losing it. He can purchase insurance of \( K \) at a cost of \( \gamma K \). Consumption in case of loss is \( c_1 = (1 - \gamma)K \) and in the case of no loss is \( c_0 = A - \gamma K \). The budget constraint for the individual is \( c_0 = A - \frac{\gamma}{1 - \gamma}c_1 \).

- An individual with income of \( m \) has a true tax liability of \( T \) but tries to evade an amount \( d \) by underdeclaration. There is probability \( \pi \) of being audited in which case he pays the full liability \( T \) plus a fine \( fD \). Consumption in case of audit is \( c_1 = m - T - fD \) and in the case of no audit is \( c_0 = m - T + D \). The budget constraint for the individual is \( c_0 = (1 + f)(m - T) - fc_1 \).

In both of these cases the budget constraint is linear and downward sloping.

Preferences are defined over quantities consumed in the different states \( (c_0, c_1, \ldots) \) and depend on perceived probabilities of the states occurring \( (\pi_0, \pi_1, \ldots) \). Under certain assumptions it may be reasonable to regard the consumer as maximising expected utility

\[ u(c_0, c_1, \ldots, \pi_0, \pi_1, \ldots) = \sum \pi_i \nu(c_i) \]
for some state-specific utility function $\nu(\cdot)$. We refer to $u(\cdot)$ as a von-Neumann-Morgenstern expected utility function.

The most controversial assumption required to justify an expected utility formulation is the strong independence axiom or sure thing principle. Consider the following two choices:

\[
\begin{array}{c|cc}
\text{Choice 1:} & \pi & 1 - \pi \\
& \text{Option } A_1 & \alpha & \gamma \\
& \text{Option } B_2 & \beta & \gamma \\
\end{array}
\]

\[
\begin{array}{c|cc}
\text{Choice 2:} & \pi & 1 - \pi \\
& \text{Option } A_2 & \alpha & \delta \\
& \text{Option } B_2 & \beta & \delta \\
\end{array}
\]

In each case the two options deliver the same outcome as each other with probability $1 - \pi$ (though these outcomes differ between the two choices). It might therefore be argued that the choice should be driven only by the different outcomes occurring with probability $\pi$. However these are the same in the two choices. Therefore if $A_1$ is preferred to $B_1$ it is argued that $A_2$ should be preferred to $B_2$. This is the sure thing principle. Combined with other less controversial axioms extending choice to uncertain situations with multiple outcomes it implies that the MRS between consumption in any two states is independent of outcomes in any other state.

The sure thing principle is violated by many people’s choices in the following example (known as the Allais paradox):

\[
\begin{array}{c|ccc}
\text{Choice 1:} & 0.10 & 0.89 & 0.01 \\
& \text{Option } A_1 & £1m & £1m & £1m \\
& \text{Option } B_1 & £5m & £1m & 0 \\
& & 0.10 & 0.89 & 0.01 \\
\end{array}
\]

\[
\begin{array}{c|ccc}
\text{Choice 2:} & 0.10 & 0.89 & 0.01 \\
& \text{Option } A_2 & £1m & 0 & £1m \\
& \text{Option } B_2 & £5m & 0 & 0 \\
\end{array}
\]

where it is common to prefer $A_1$ to $B_1$ and $B_2$ to $A_2$.

Note that the function $\nu(\cdot)$ is not ordinal. Preferences are changed by arbitrary increasing transformations of $\nu(\cdot)$. However $u(\cdot)$ is still ordinal.

**Risk aversion**

To capture risk aversion we need to capture the fact that risk averse individuals prefer to receive the expected value of any gamble with certainty to undertaking the gamble. Thus

$$\nu((1-\pi)c_0 + \pi c_1) > (1-\pi)\nu(c_0) + \pi \nu(c_1).$$

For this always to be true requires that $\nu(\cdot)$ be a concave function. The degree of concavity is an indicator of the strength of aversion to risk.
Consider the insurance case again. A utility maximising consumer chooses $K$ to maximise

$$(1 - \pi)\nu(A - \gamma K) + \pi\nu((1 - \gamma)K).$$

The first order condition requires

$$(1 - \pi)\gamma\nu'(A - \gamma K) = (1 - \gamma)\pi\nu'((1 - \gamma)K).$$

If insurance is actuarially fair then $\pi = \gamma$ and therefore $\nu'(A - \gamma K) = \nu'((1 - \gamma)K)$. If the individual is risk averse then $\nu'(\cdot)$ is a decreasing function and therefore $A = K$ i.e. there is full insurance. This is a typical illustration of behaviour under risk. The fairness of insurance means that risk can be eliminated without compromising expected consumption and a risk averse individual chooses therefore to eliminate risk.

If insurance is less than fair $\pi < \gamma$ then there is underinsurance.

**Equilibrium in pure exchange economies**

Suppose the $h$th household has endowment $\omega^h$ and consumes a bundle $q^h$. An allocation of goods is said to be *feasible* if the aggregate amount consumed of each good equals the aggregate endowment

$$\sum_h q^h_i = \sum_h \omega^h_i \quad i = 1, 2, ...$$

The initial endowments obviously constitute one feasible allocation.

Demands if prices are $p$ are $q^h_i = f^h_i(p'\omega^h, p)$, $i = 1, 2, ...$ where $p'\omega^h$ is the value of the individual’s endowment. Market demand is found by adding the demands across individuals

$$Q_i(p'\omega^1, p'\omega^2, ..., p) = \sum_h f^h_i(p'\omega^h, p) \quad i = 1, 2, ...$$

Note the dependence on the complete distribution of endowments.

Let $z^h_i$ denote the excess demand from the $i$th household. The aggregate excess demand $z_i$ is given by the excess of market demand over the sum of endowments

$$z_i(p'\omega^1, p'\omega^2, ..., p) = \sum_h z^h_i = \sum_h [f^h_i(p'\omega^h, p) - \omega^h_i] \quad i = 1, 2, ...$$

**General equilibrium** - referred to as *market equilibrium*, *competitive equilibrium* or *Walrasian equilibrium* - is a set of prices such that aggregate excess demand is zero on all markets.

$$\sum_h z^h_i = 0 \quad i = 1, 2, ...$$

The competitive allocation is another example of a feasible allocation.
If there are \( k \) goods then this seems to define \( k \) equations in \( k \) unknown prices. However this is misleading. The fact that each household must be on its budget constraint implies that the value of that household’s excess demand is zero

\[
\sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad \Rightarrow \quad \sum_i p_i z_i^h = 0.
\]

Adding this equation over households establishes that the value of aggregate excess demand is also zero

\[
\sum_i p_i \sum_h z_i^h = 0.
\]

This is *Walras’ law* and is true for any prices (not only the equilibrium prices). It implies that the \( k \) excess demands are not independent - in fact there are only \( k - 1 \) independent excess demands to set to zero.

However since demands are homogeneous multiplying all prices by any positive number will give the same excess demands. If any prices constitute a Walrasian equilibrium, then so therefore do any positive multiple of those prices. It is therefore only relative prices which are determined by the equilibrium conditions.

There are therefore actually \( k - 1 \) independent equations determining \( k - 1 \) relative prices.

Nothing said so far ensures existence of a Walrasian equilibrium but if aggregate demands vary continuously as a function of prices then we may expect this to be so.

### Efficiency

Walrasian equilibrium has the general property of *Pareto efficiency*. This means that there is no feasible allocation such that all consumers are better off (or some are better off without any being any worse off). To prove this suppose it were not the case. Then there would exist an allocation \( r^1, r^2, \ldots \) such that \( r^1 \) was preferred to \( q^1 \), \( r^2 \) was preferred to \( q^2 \) and so on. But then these bundles could not be affordable at the equilibrium prices \( p \) or the consumers would have purchased them. Thus

\[
\sum_i p_i r_i^h > \sum_i p_i q_i^h = \sum_i p_i \omega_i^h \quad h = 1, 2, \ldots
\]

Adding across consumers gives

\[
\sum_i p_i \sum_h r_i^h > \sum_i p_i \sum_h \omega_i^h.
\]

But this conflicts with feasibility which requires \( \sum_h r_i^h = \sum_h \omega_i^h, \; i = 1, 2, \ldots \) Hence there can be no such alternative allocation.

This is the *First Fundamental Theorem of Welfare Economics*. Walrasian equilibrium is always Pareto efficient. Note that this says nothing about desirability in other respects such as distributional equity - this, for instance, will inevitably depend upon the equity in the allocation of initial endowments.
The Second Fundamental Theorem of Welfare Economics tells us conversely that, under certain further assumptions, any Pareto efficient allocation can be sustained as a Walrasian equilibrium given the right allocation of initial endowments. In particular, this is true if we assume all agents have convex preferences.

Solving general equilibrium problems for pure exchange economies

Suppose the $h$th household has endowment $\omega^h$ and consumes a bundle $q^h$. Demands if prices are $p$ are $q^h_i = f^h_i(y^h, p)$, $i = 1, 2, ..., k$ where $y^h = p'\omega^h$ is the value of the individual’s endowment. We need to find a price vector $p$ solving the market clearing equations

$$\sum_h f^h_i(p'\omega^h, p) = \sum_h \omega^h_i \quad i = 1, 2, ..., k$$

From Walras’ law we need only solve for $k - 1$ relative prices achieving market clearing on $k - 1$ of the $k$ markets.

As an example, suppose there are two goods. We can solve only for the relative price so we normalise the price of good 2 to be 1 and let $p$ be the price of the first. Let the two consumers have Cobb Douglas demands over the two goods. Individual $h$ therefore has demands

$$q^h_1 = \alpha^h y^h / p \quad q^h_2 = (1 - \alpha^h) y^h$$

where $\alpha^h$ is an individual-specific taste parameter.

Individual endowments are $\omega^h = (\omega^h_1, \omega^h_2)$. Therefore, by substituting the value of the endowments, demands are

$$q^h_1 = \alpha^h (p\omega^h_1 + \omega^h_2) / p \quad q^h_2 = (1 - \alpha^h)(p\omega^h_1 + \omega^h_2)$$

To find equilibrium, we know from Walras’ law that we need only find the price to clear one market. Take the first. Market clearing requires

$$\omega^1_1 + \omega^2_1 = q^1_1 + q^2_1 = \alpha^1 (p\omega^1_1 + \omega^2_1) / p + \alpha^2 (p\omega^2_1 + \omega^2_2) / p$$

Solving for $p$ gives

$$p = \frac{\alpha^1 \omega^1_1 + \alpha^2 \omega^2_2}{(1 - \alpha^1)\omega^1_1 + (1 - \alpha^2)\omega^2_1}.$$ 

Note that this is increasing in endowments of the second good and decreasing in endowments of the first. Note also that it is increasing in the demand parameters $\alpha^1$ and $\alpha^2$. These are readily intelligible demand and supply effects for this example.