HETEROCLINIC LIMIT CYCLES IN COMPETITIVE KOLMOGOROV SYSTEMS

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Abstract. A notion of global attraction and repulsion of heteroclinic limit cycles is introduced for strongly competitive Kolmogorov systems. Conditions are obtained for the existence of cycles linking the full set of axial equilibria and their global asymptotic behaviour on the carrying simplex. The global dynamics of systems with a heteroclinic limit cycle is studied. Results are also obtained for Kolmogorov systems where some components vanish as $t \to \pm \infty$.

1. Introduction. Consider the Kolmogorov system

$$x'_i = x_i f_i(x), \quad i \in I_N := \{1, 2, \ldots, N\}, \ x \in \mathbb{R}_+^N,$$  

where $f = (f_1, \ldots, f_N)^T : \mathbb{R}_+^N \to \mathbb{R}^N$ is at least $C^1$. Since we are mainly dealing with a special class of system (1) that are strongly competitive (to be explained below), we assume that $f_i(0) > 0$ for $i \in I_N$ so that the origin is a repeller. Let $e_i = (0, \ldots, 1, \ldots, 0)^T$, where the ‘1’ is in the $i$th position. We also assume that $f_i(e_i) = 0$ so that $e_i$ is an axial fixed point for each $i \in I_N$. Moreover, $e_1, \ldots, e_N$ are assumed to be the only axial fixed points. We restrict the study of (1) to the invariant sets $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : \forall i \in I_N, x_i \geq 0\}$, its interior $\text{int} \mathbb{R}_+^N = \{x \in \mathbb{R}^N : \forall i \in I_N, x_i > 0\}$ and its boundary $\partial \mathbb{R}_+^N = \mathbb{R}_+^N \setminus \text{int} \mathbb{R}_+^N$.

Heteroclinic cycles occur in many dynamical systems that are models of physical or biological systems (for example [2, 6, 17]). For general dynamical systems with symmetry, a theory for asymptotic stability, structural stability and various bifurcations of heteroclinic cycles have been established (see, for example, [9] and the more recent paper by the same authors [10] and references therein). Some of the deep results established for systems with symmetry can also be applied to systems, such as (1), that do not possess symmetry (for example, see remark 2.8 of [9]). However, we have not been able to find results for global stability of heteroclinic cycles for either of the symmetry or non-symmetric cases.

We denote the solution of a system with the initial condition $x(0) = x^0$ by $x(t, x^0)$ and define its $\alpha$-limit set, $\omega$-limit set by $\alpha(x^0) = \cap_{T \geq 0} \overline{\{x(t, x^0) : t \leq T\}}$, $\omega(x^0) = \cap_{T \geq 0} \{x(t, x^0) : t \geq T\}$ respectively, where the overline denotes set closure.

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Here we are concerned with heteroclinic cycles of (1) and in particular when they attract or repel certain subsets of \( \mathbb{R}^N_+ \). By a heteroclinic cycle we mean a closed curve that is topologically a circle consisting of fixed points \( p_i \) for \( i \in I_m \), \( m \geq 2 \) together with heteroclinic trajectories \( T_i \) that connect \( p_i \) to \( p_{i+1} \) (here \( p_{m+1} = p_1 \)).

By a heteroclinic limit cycle \( \Gamma \) we mean a heteroclinic cycle \( \Gamma \) with an attracting (or repelling) neighbourhood \( N(\Gamma) \) (restricted to \( \text{int}\mathbb{R}^N_+ \) or \( \mathbb{R}^N_+ \)) such that \( \omega(x^0) = \Gamma \) (or \( \alpha(x^0) = \Gamma \)) for all \( x^0 \in N(\Gamma) \). The main issue we address here is when the heteroclinic cycle is \textit{globally} attracting or repelling (in some sense defined below).

Whereas it is straightforward to define global stability for a heteroclinic limit cycle, to define global repulsion is intricate as it is not simply by reversing the time in \( \text{int}\mathbb{R}^N_+ \). Our approach here is restricted to strongly competitive (1), where it is known that all limit sets belong to an invariant hypersurface known as the carrying simplex.

Strongly competitive Kolmogorov systems are characterised by \( \frac{\partial f}{\partial x} < 0 \) for all \( i, j \in I_N \). For such systems the results of Hirsch [11] guarantee the existence of a \textit{carrying simplex}, a Lipschitz invariant manifold \( \Sigma \) of dimension \( N - 1 \) that attracts all of \( \mathbb{R}^N_+ \setminus \{0\} \) as \( t \to +\infty \). Moreover, it is known that every trajectory of (1) in \( \mathbb{R}^N_+ \setminus \{0\} \) is asymptotic to one in the carrying simplex \( \Sigma \) as \( t \to +\infty \). We are concerned with the global dynamics of the system when it has an attracting (repelling) heteroclinic limit cycle in the carrying simplex \( \Sigma \). When \( N = 3 \), it is known that the dynamics on \( \Sigma \) is topologically equivalent to that of a planar system.

Strongly competitive Lotka-Volterra systems are a subclass of these systems, for which each \( f_i \) is a linear function, so that they assume the form
\[ x_i' = r_i x_i (1 - A_i x), \quad i \in I_N, \tag{2} \]
where \( A_i = (a_{i1} \cdots a_{iN}) \) is the \( i \)th row of a matrix \( A = (a_{ij}) \) and the \( r_i \) and \( a_{ij} \) are positive constants with \( a_{ii} = 1 \).

A well-known and illustrative example of (2) is the May-Leonard [16] system
\[
\begin{align*}
x_1' &= x_1(1 - x_1 - \alpha_1 x_2 - \beta_1 x_3), \\
x_2' &= x_2(1 - \beta_2 x_1 - x_2 - \alpha_2 x_3), \\
x_3' &= x_3(1 - \alpha_3 x_1 - \beta_3 x_2 - x_3)
\end{align*}
\tag{3}
\]
with \( 0 < \alpha_i < 1 < \beta_i \) for \( i \in I_3 \). This was first studied for the case \( \alpha_i = \alpha, \beta_i = \beta \) in [16]. The more general model has been treated in [5] and [27]. Hirsch’s results show that this strongly competitive system has a 2-dimensional carrying simplex \( \Sigma \) and that all points except the origin are attracted onto it. The present parameter ranges \( \Sigma \) contains 3 axial fixed points plus a unique interior fixed point \( p \). In [5] it was shown that when \( \prod_{i=1}^3 (1 - \alpha_i) > \prod_{j=1}^3 (\beta_j - 1) \) the interior fixed point \( p \) is globally asymptotically stable and there is a heteroclinic cycle \( \Gamma_0 \) such that \( \alpha(x^0) = \Gamma_0 \) for all \( x^0 \in \text{int}\Sigma \setminus \{p\} \), and that when \( \prod_{i=1}^3 (1 - \alpha_i) < \prod_{j=1}^3 (\beta_j - 1) \) the interior fixed point is globally repelling on \( \Sigma \) and \( \omega(x^0) = \Gamma_0 \) for all \( x^0 \in \text{int}\Sigma \setminus \{kp : k > 0\} \).

The heteroclinic cycle is \( e_1 \to e_3 \to e_2 \to e_1 \), and is reversed if \( 0 < \beta_i < 1 < \alpha_i \) and the same other conditions hold.

Hofbauer and Sigmund partially extended these results to the more general Lotka-Volterra system
\[
\begin{align*}
x_1' &= r_1 x_1(1 - x_1 - \alpha_2 x_2 - \beta_3 x_3), \\
x_2' &= r_2 x_2(1 - \beta_1 x_1 - x_2 - \alpha_3 x_3), \\
x_3' &= r_3 x_3(1 - \alpha_1 x_1 - \beta_2 x_2 - x_3)
\end{align*}
\tag{4}
\]
with \( r_i > 0 \) and \( \beta_i < 1 < \alpha_i \) (the \( \beta_i \) are not constrained to be positive) [12]. They proved that (4) is permanent (i.e. \( \exists M_0 > 0, \forall t \geq M_1 > 0 \) such that all solutions in \( \text{int}R^+_N \) satisfy \( M_0 < x_i(t) < M_1 \) for all \( i \in I_3 \) and large \( t \)) if \( \det A > 0 \) and \( \prod_{i=1}^3 (\alpha_i - 1) < \prod_{j=1}^3 (1 - \beta_j) \), and if \( \prod_{i=1}^3 (\alpha_i - 1) > \prod_{j=1}^3 (1 - \beta_j) \) then (4) has a (locally) attracting heteroclinic cycle \( e_1 \to e_2 \to e_3 \to e_1 \). Necessary and sufficient conditions for global stability or repulsion on \( \Sigma \) are, to the best of our knowledge, not known.

For the \( N \)-dimensional system (2) with
\[
\forall i \in I_N, \forall k \in I_N \setminus \{i, i + 1\}, \quad r_i = 1, \quad a_{(i+1)i} < 1, \quad a_{ki} > 1, \tag{5}
\]
where \( N + 1 = 1 \mod N \), using Poincaré maps Afraimovich et al. [1] showed that the heteroclinic cycle \( \Gamma_0 : e_1 \to e_2 \to \cdots \to e_N \to e_1 \) connecting the \( N \) axial fixed points is stable in the sense that \( \lim_{t \to +\infty} \text{dist}(x(t, x^0), \Gamma_0) = 0 \) when \( x^0 \) is in a neighborhood \( U \) of \( \Gamma_0 \) in \( \text{int}R^+_N \), provided that
\[
\Omega(A) := \prod_{i=1}^N \frac{a_{i(i+1)} - 1}{1 - a_{(i+1)i}} > 1.
\]
Indeed, an earlier result dealing with more general systems was given by Feng [7]. For (2) satisfying
\[
\forall i \in I_N, \forall j \in I_N \setminus \{i, i + 1\}, \quad a_{ji} > 1, \quad 0 < a_{(i+1)i} < 1
\]
or
\[
\forall i \in I_N, \forall j \in I_N \setminus \{i - 1, i\}, \quad a_{ji} > 1, \quad 0 < a_{(i-1)i} < 1,
\]
where \( N = 0 \) and \( N + 1 = 1 \mod N \), a sufficient condition similar to the one given in [1] was provided in [7] for the stability of the heteroclinic cycle \( \Gamma_0 \). Based on the result given in [1], Hsu and Roeger [15] also obtained conditions for existence and stability of a heteroclinic cycle in a chemostat model.

Following remark 2.8 of [9], which deals principally with heteroclinic cycles in systems with symmetry but also includes results non-symmetric systems, all of the above examples can also be treated using Theorem 2.7 of that same paper.

Hofbauer gives a concise way of analysing heteroclinic cycles in ecological systems via a characteristic matrix [13] based upon average Lyapunov functions, which are particularly well-suited to ecological systems. Field and Swift [8] use Poincaré maps and transition matrices to study 4 dimensional system with symmetry. Let \( \Gamma \) be a heteroclinic cycle consisting of fixed points \( \omega_k, k = 1, \ldots, m, \) and lying in \( \partial R^+_N \).

Then Hofbauer constructs the \( m \times N \) characteristic matrix \( C \) with \( c_{ij} = f_j(\omega_i) \) if \( x_i = 0 \) at \( \omega_j \) and \( c_{ij} = 0 \) if \( x_i > 0 \) at \( \omega_j \). Let us introduce the standard notation: For any two vectors \( p \) and \( q, p \gg q (p \leq q) \) means that \( p_i > q_i \) \( (p_i < q_i \) for each \( i \in I_N \). Then, (a) if there is an \( h \gg 0 \) such that \( Ch \gg 0 \) then \( \partial R^+_N \) is repelling (i.e. trajectories in \( \text{int}R^+_N \) move away from \( \partial R^+_N \) near \( \Gamma \)); (b) if \( \Gamma \) is asymptotically stable in \( \partial R^+_N \), and there is an \( h \ll 0 \) such that \( Ch \ll 0 \), then \( \Gamma \) is asymptotically stable in \( R^+_N \). For example, for the system (4) we obtain (up to multiplication by a permutation matrix)
\[
C = \begin{pmatrix}
0 & r_2(1 - \beta_1) & r_3(1 - \alpha_1) \\
r_1(1 - \alpha_2) & 0 & r_3(1 - \beta_2) \\
r_1(1 - \beta_3) & r_2(1 - \alpha_3) & 0
\end{pmatrix}.
\]
With $h = (h_1, h_2, h_3)^T$ we obtain

$$Ch = \begin{pmatrix} r_2(1 - \beta_1)h_2 + r_3(1 - \alpha_1)h_3 \\ r_1(1 - \alpha_2)h_1 + r_3(1 - \beta_2)h_3 \\ r_1(1 - \beta_3)h_1 + r_2(1 - \alpha_3)h_2 \end{pmatrix}. $$

Then it can be shown that there is a $h \gg 0$ with $Ch \gg 0$ if and only if \( \prod_{j=1}^3 (\alpha_i - 1) < \prod_{j=1}^3 (1 - \beta_j) \), whereas there is a $h \ll 0$ with $Ch \ll 0$ if and only if $\prod_{i=1}^3 (\alpha_i - 1) > \prod_{j=1}^3 (1 - \beta_j)$. The other examples listed above may also be studied using their characteristic matrices.

We observe that in all these examples and to the best of our knowledge elsewhere in the literature, the concept of stability of a heteroclinic cycle deals only with the local behaviour near this cycle. We also notice that the stability concept of a heteroclinic cycle is distinct from the concept of attracting or repelling heteroclinic limit cycle.

**Definition 1.1.** We say that a heteroclinic cycle $\Gamma_0$ of (1) is a

- locally attracting (repelling) heteroclinic limit cycle if there is a neighbourhood $V \subset \text{int} \mathbb{R}^N_+$ of $\Gamma_0$ such that $\omega(x^0) = \Gamma_0$ ($\alpha(x^0) = \Gamma_0$) for all $x^0 \in V$;
- globally attracting (repelling) heteroclinic limit cycle if $\omega(x^0) = \Gamma_0$ ($\alpha(x^0) = \Gamma_0$) for all $x^0 \in \text{int} \mathbb{R}^N_+ \setminus U$ ($x^0 \in \text{int} \Sigma \setminus U$), where $U$ is a union of a finite number of manifolds of dimension lower than $N (N - 1)$ and the set of fixed points.

In summary, our main interest is in the global dynamics of the strongly competitive system (1). Even if we know the existence and local stability (repulsion) of a heteroclinic cycle and the local behaviour near each fixed point, the global dynamics could still be very complicated and not deducible from knowledge of this local stability. For example, even for $N = 3$, an unstable interior fixed point and an unstable heteroclinic cycle could contain many (but finite in number) nested limit cycles [23]. Possibilities of complicated behaviour (for $N \geq 4$) includes the existence of a strange attractor within $\Sigma$, chaotic behaviour, and so on. In some circumstances, however, it is also possible that the global dynamics is not so complicated and the global behaviour is simply an extension of local behaviour, e.g. a locally attracting (repelling) heteroclinic cycle is actually globally attracting (repelling). In this paper, instead of tackling the complexity of the global dynamics, we aim at excluding the possibility of complicated global behaviour by exploring conditions under which (1) has a globally attracting (repelling) heteroclinic limit cycle.

We denote the $i$th coordinate plane by $\pi_i = \{ x \in \mathbb{R}^N : x_i = 0 \}$ and the $i$th nullcline by $\gamma_i = \{ x \in \mathbb{R}^N : f_i(x) = 0 \}$. A set $S$ is said to be below (above) $\gamma_i$ if $f_i(x) \geq 0 \ (\leq 0)$ for all $x \in S$, and strictly below (strictly above) $\gamma_i$ if $f_i(x) > 0 \ (< 0)$ for all $x \in S$.

Most of the results presented in this paper for attracting heteroclinic limit cycles are parallel to those for repelling heteroclinic limit cycles. To save some space, we adopt the style of merging the parallel statements into one with one set of the parallel words in brackets, e.g. “attracting (repelling)”.

**2. Heteroclinic limit cycles in three-dimensional systems.** In this section, we consider system (1) with $N = 3$:

$$x'_i = x_i f_i(x), \ i \in I_3. $$

(6)
Assuming that there is a unique interior globally attracting (repelling on $\Sigma$) fixed point, we explore necessary as well as sufficient conditions for (1) to have a globally repelling (attracting) heteroclinic limit cycle, as defined by Definition 1.1.

We emphasize that when we say $p \in \text{int}\Sigma$ is globally attracting (repelling) we mean relative to $\text{int}\Sigma$. However, when $p \in \text{int}\Sigma$ is globally attracting relative to $\text{int}\Sigma$, it is also globally attracting relative to $\text{int}\mathbb{R}^N$.

**Theorem 2.1.** Suppose that the Kolmogorov system (6) has a finite number of fixed points and is strongly competitive: $\frac{\partial f_i}{\partial x_j} < 0$ for $x \in \mathbb{R}^3_+$ with $x_i > 0$. Assume that $\partial \Sigma$ is a heteroclinic limit cycle. Then, with $4 = 1 \pmod{3}$ and $5 = 2 \pmod{3}$, we have either (7) or (8):

\[
\forall i \in I_3, \quad \gamma_i \cap \pi_{i+2} \text{ is below } \gamma_{i+1} \cap \pi_{i+2}, \quad (7)
\]

\[
\forall i \in I_3, \quad \gamma_i \cap \pi_{i+2} \text{ is above } \gamma_{i+1} \cap \pi_{i+2}. \quad (8)
\]

Moreover, with $E = \{e_1, e_2, e_3\}$, if $q \in \partial \Sigma \setminus E$ is a fixed point, then, for any $j \in I_3$, $q_j = 0$ implies $f_j(q) = 0$.

**Proof.** Suppose neither (7) nor (8) holds. Then there are only the following two cases:

(i) For some distinct $i, j \in I_3$, $\gamma_i \cap \pi_{i+2}$ is below $\gamma_{i+1} \cap \pi_{i+2}$ but $\gamma_j \cap \pi_{j+2}$ is above $\gamma_{j+1} \cap \pi_{j+2}$.

(ii) For some $i \in I_3$, $\gamma_i \cap \pi_{i+2}$ is neither below nor above $\gamma_{i+1} \cap \pi_{i+2}$.

In case (i), by a permutation of $(1, 2, 3)$ if necessary, we may assume that $\gamma_1 \cap \pi_3$ is below $\gamma_2 \cap \pi_3$ but $\gamma_2 \cap \pi_1$ is above $\gamma_3 \cap \pi_1$. Since (6) has only a finite number of fixed points, a phase portrait on $\pi_3$ shows that the flow direction on $\pi_3 \cap \Sigma$ is from $e_1$ to $e_2$ whereas a phase portrait on $\pi_1$ shows that the flow direction on $\pi_1 \cap \Sigma$ is from $e_3$ to $e_2$, a contradiction to $\partial \Sigma$ being a heteroclinic cycle.

In case (ii), we may assume that there are $x, y \in \gamma_1 \cap \pi_3$ with $x_1 < y_1$ such that $f_2(x) < 0$ but $f_2(y) > 0$. Since (6) has only a finite number of fixed points and $f$ is continuous, there is a $q \in \gamma_1 \cap \gamma_2 \cap \pi_3$ with $q_1 > 0$ and $q_2 > 0$ such that for $z \in \gamma_1 \cap \pi_3$ close enough to $q$, $f_2(z) < 0$ if $z_1 < q_1$ but $f_2(z) > 0$ if $z_1 > q_1$. Then a phase portrait on $\pi_3$ shows that $q$ is a saddle point, a contradiction to $\partial \Sigma$ being a heteroclinic cycle.

The above contradictions confirm that either (7) or (8) holds.

If $q \in \partial \Sigma \setminus E$ is a fixed point such that $q_j = 0$ but $f_j(q) \neq 0$ for some $j \in I_3$, then $f_j(q)$ is an eigenvalue of the linearised system at $q$ with an eigenvector pointing to or from $\text{int}\mathbb{R}^3_+$ so there is a point $x^0 \in \text{int}\mathbb{R}^3_+$ such that $\alpha(x^0) = \{q\}$ or $\omega(x^0) = \{q\}$. This contradicts the assumption that $\partial \Sigma$ is a heteroclinic limit cycle. Therefore, $q_j = 0$ must imply $f_j(q) = 0$.

**Corollary 1.** Assume that the Lotka-Volterra system (2) with $N = 3$ has a finite number of fixed points and that $\partial \Sigma$ is a heteroclinic cycle. Then we have either (9) or (10):

\[
\forall i \in I_3, \quad \gamma_i \cap \pi_{i+2} \setminus E \text{ is strictly below } \gamma_{i+1} \cap \pi_{i+2} \setminus E, \quad (9)
\]

\[
\forall i \in I_3, \quad \gamma_i \cap \pi_{i+2} \setminus E \text{ is strictly above } \gamma_{i+1} \cap \pi_{i+2} \setminus E. \quad (10)
\]

Moreover, $e_1, e_2$ and $e_3$ are the only fixed points on $\partial \Sigma$.

The following lemma will be required in the sequel:
Lemma 2.2. (Butler-McGhee [19, 4]) Suppose that \( p \) is a hyperbolic fixed point of an autonomous system \( y' = g(y) \), \( p \in \omega(x^0) \) but \( \{p\} \neq \omega(x^0) \) for some \( x^0 \). Let \( W^s(p) \) be the stable manifold of \( p \) and \( W^u(p) \) the unstable manifold. Then \( \omega(x^0) \cap (W^s(p) \setminus \{p\}) \neq \emptyset \) and \( \omega(x^0) \cap (W^u(p) \setminus \{p\}) \neq \emptyset \).

Note that Lemma 2.2 is still valid after the replacement of \( \omega(x^0) \) by \( \alpha(x^0) \) since \( \omega(x^0) \) of \( y' = g(y) \) becomes \( \alpha(x^0) \) of \( y' = -g(y) \).

Theorem 2.3. Assume that the Kolmogorov system (6) is strongly competitive and satisfies the following conditions:

(a) There is a unique fixed point \( p \) in \( \text{int}\mathbb{R}^3_+ \) that is globally asymptotically stable (hyperbolic with one-dimensional stable manifold \( W^s(p) \) in \( \text{int}\mathbb{R}^3_+ \) and globally repelling on \( \Sigma \)).

(b) On \( \partial \Sigma \) (6) has only three fixed points \( e_1, e_2, e_3 \) and either the inequalities (11) or (12) hold:

\[
\forall i \in I_3, \quad f_i(e_{i+1}) < 0 < f_{i+2}(e_{i+1}), \tag{11}
\]

\[
\forall i \in I_3, \quad f_i(e_{i+1}) > 0 > f_{i+2}(e_{i+1}). \tag{12}
\]

Then \( \partial \Sigma \) is a globally repelling (attracting) heteroclinic limit cycle.

Proof. First, from assumption (b) it is clear that \( \partial \Sigma \) is a closed curve containing all boundary fixed points \( e_1, e_2, e_3 \). From (11) or (12) and phase portraits on \( \pi_1, \pi_2, \pi_3 \) we see that the flow on \( \partial \Sigma \) as \( t \) increases is shown either in Figure 1(a) or Figure 1(b).

Thus, \( \partial \Sigma \) is a heteroclinic cycle. It therefore remains to be proved that \( \partial \Sigma \) is globally repelling (attracting). For each \( x^0 \in \text{int}\Sigma \setminus \{p\} \) \((x^0 \in \text{int}\mathbb{R}^3_+ \setminus W^s(p))\), the assumption (a) ensures that \( \alpha(x^0) \subset \partial \Sigma \ (\omega(x^0) \subset \partial \Sigma) \). Since \( \alpha(x^0) \) (\( \omega(x^0) \)) is nonempty, connected, compact and invariant, it contains at least one boundary fixed point, say \( e_1 \). The Jacobian of the system is

\[
J(x) = \begin{pmatrix}
  f_1(x) + x_1 \frac{\partial f_1}{\partial x_1} & x_1 \frac{\partial f_1}{\partial x_2} & x_1 \frac{\partial f_1}{\partial x_3} \\
  x_2 \frac{\partial f_2}{\partial x_1} & f_2(x) + x_2 \frac{\partial f_2}{\partial x_2} & x_2 \frac{\partial f_2}{\partial x_3} \\
  x_3 \frac{\partial f_3}{\partial x_1} & x_3 \frac{\partial f_3}{\partial x_2} & f_3(x) + x_3 \frac{\partial f_3}{\partial x_3}
\end{pmatrix}.
\]

From this and (11) or (12) we see that the boundary fixed points are hyperbolic and their stable and unstable manifolds are all subsets of \( \partial \mathbb{R}^3_+ \). Since \( x^0 \) belongs to neither the stable nor the unstable manifold of \( e_1 \), we have \( e_1 \in \alpha(x^0) \) \((e_1 \in \omega(x^0))\).
but \( \{e_1\} \neq \alpha(x^0) \) \((\{e_1\} \neq \omega(x^0))\). By Lemma 2.2 and the invariance of \(\alpha(x^0)\) \((\omega(x^0))\), we obtain \(T_3 \subset \alpha(x^0)\) \((T_3 \subset \omega(x^0))\) and \(T_3 \subset \alpha(x^0)\) \((T_1 \subset \omega(x^0))\). By the compactness of \(\alpha(x^0)\) \((\omega(x^0))\), we must have \(e_2, e_3 \in \alpha(x^0)\) \((e_2, e_3 \in \omega(x^0))\). As \(x^0\) belongs to none of the stable and unstable manifolds of \(e_2\) and \(e_3\), by Lemma 2.2 again, we also have \(T_2 \subset \alpha(x^0)\) \((T_2 \subset \omega(x^0))\). Therefore, we have proved \(\alpha(x^0) = \partial\Sigma\) \((\omega(x^0) = \partial\Sigma)\). Hence, \(\partial\Sigma\) is globally repelling (attracting). \(\square\)

### 2.1. Example 1

Consider the system

\[
\begin{align*}
x'_1 &= x_1 f_1(x) = x_1 \left(1 - x_1 - \beta \frac{x_2}{1 + x_3} - \alpha x_3 \right), \\
x'_2 &= x_2 f_2(x) = x_2 \left(1 - \alpha x_1 - x_2 - \beta \frac{x_3}{1 + x_1} \right), \\
x'_3 &= x_3 f_3(x) = x_3 \left(1 - \beta \frac{x_1}{1 + x_2} - \alpha x_2 - x_3 \right),
\end{align*}
\]

where \(\alpha > 0\) and \(\beta > 0\). Since each \(\pi_i\) is invariant and \(x'_i < 0\) for \(x \in \mathbb{R}^3\) with \(x_i = 1\) but \(x \neq e_i\), \(B = [0, 1]^3\) is forward invariant for this system and \(\omega(x^0) \subset B\) for every \(x^0 \in \mathbb{R}^3\). Since

\[
Df = \begin{pmatrix}
-1 & -\frac{\beta x_2}{(x_3+1)^2} - \alpha & \frac{\beta x_2}{(x_3+1)^2} - \alpha \\
\frac{\beta x_2}{(x_3+1)^2} - \alpha & -\frac{\beta}{x_2+1} & \frac{\beta}{x_2+1} \\
\frac{\beta}{x_2+1} & \frac{\beta}{x_2+1} - \alpha & -1
\end{pmatrix},
\]

for \(\alpha > \beta\) the system is strongly competitive in \(B\). Apart from the axial fixed points \(e_1, e_2, e_3\), there are no other boundary fixed points when \(\alpha > 1, \beta < 1\). There is at least one interior fixed point \(p = \mu(1, 1, 1)^T\) where \(\mu = \frac{\sqrt{2\alpha\beta+(\alpha+2)^2+\beta^2} - \alpha - \beta}{2(\alpha+1)}\) (we shall show there are no other fixed points later). Let \(S = x_1 + x_2 + x_3\) and \(P = x_1 x_2 x_3\). Following the method on page 51 of [14], we define \(V = PS^{-3}\). Then

\[
V' = V \left(3 - (1 + \alpha)S - \frac{\beta x_1}{1 + x_2} - \frac{\beta x_2}{1 + x_3} - \frac{\beta x_3}{1 + x_1} - 3(x'_1 + x'_2 + x'_3) - \frac{3}{S} \right)
\]

\[
= \frac{(2 - \alpha)V}{2S} \left\{(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right\} + \frac{\beta V}{S} T,
\]

where

\[
T = \frac{x_3}{1 + x_1} (2x_2 - x_1 - x_3) + \frac{x_1}{1 + x_2} (2x_3 - x_1 - x_2) + \frac{x_2}{1 + x_3} (2x_1 - x_2 - x_3).
\]

The first term in \(V'\) is negative for \(\alpha > 2\) when \(x \neq s(1, 1, 1)^T\) for any \(s \geq 0\). We shall show that if \(\alpha > 2\) then every solution of (13) in \(\text{int}\mathbb{R}^3_+\) satisfies \(S < 1\) for large \(t\) so that \(\text{int}\Sigma\) is strictly below the plane \(S = 1\). Then we shall see that \(T \leq 0\) for any \(S < 1\).

When \(\alpha > 2\),

\[
S' = S(1 - S) + x_1 x_2 \left(2 - \alpha - \frac{\beta}{1 + x_3} \right)
\]

\[
+ x_1 x_3 \left(2 - \alpha - \frac{\beta}{1 + x_2} \right) + x_2 x_3 \left(2 - \alpha - \frac{\beta}{1 + x_1} \right) < S(1 - S). \]

This shows that if $S \geq 1$ and $x^0$ is not on any $x_i$-axis then $S$ is strictly decreasing so $x(t, x^0)$ satisfies $S < 1$ for large enough $t$. Then, for $S < 1$,

\[(1 + x_1)(1 + x_2)(1 + x_3)T \]

\[= x_3(2x_2 - x_1 - x_3)(1 + x_2)(1 + x_3) + x_1(2x_3 - x_1)
- x_2)(1 + x_1)(1 + x_3) + x_2(2x_1 - x_2 - x_3)(1 + x_1)(1 + x_2)
\]

\[= -\frac{1}{2}(1 - x_2^2 - x_3^2)(x_1 - x_2)^2 + (1 - x_1^2 - x_2^2)(x_1 - x_3)^2
+ (1 - x_1^2 - x_3^2)(x_2 - x_3)^2 - \frac{1}{4}(x_1^2 - x_2^2)^2 + (x_1^2 - x_2^2)^2
+ (x_2^2 - x_3^2)^2 - x_1(x_1 - x_2)(x_1 - x_3)
+ x_3(x_2 - x_1)(x_2 - x_3) + x_3(x_3 - x_1)(x_3 - x_2)].

Note that $S < 1$ for $x \in B$ implies $x_i^2 \leq x_i < 1$ for $i \in I_3$ so the value in each of the first two pairs of square brackets above is nonnegative. Denote the value in the third pair of square brackets by $h(x_1, x_2, x_3)$. Then $h$ is invariant under any permutation on the subscripts $(1, 2, 3)$. Thus, without loss of generality, we may assume that $0 \leq x_1 \leq x_2 \leq x_3$. Then

\[h = [x_1(x_2 - x_1)(x_3 - x_1) + x_2(x_2 - x_3)^2 + (x_3 - x_1)(x_2 - x_3)^2] \geq 0.
\]

Therefore, $S < 1$ for $x \in B$ implies $T \leq 0$.

Note that the set $V_c$ of $x$ satisfying $V(x) = c$ for each $c \in \left(0, \frac{1}{27}\right)$ is a cone consisting of rays starting from the origin with $V_0 = \partial \mathbb{R}_+^3$ and $V_{1/27} = \{s(1, 1, 1)^T : 0 \leq s < +\infty\}$. Therefore, $\alpha > 2$ ensures that $p$ is a global repeller on $\Sigma$ and $\omega(x^0) \subset \partial \Sigma$ for every $x^0 \in \text{int} \mathbb{R}_+^3 \setminus \{s(1, 1, 1)^T : 0 \leq s < +\infty\}$. From (13) we have

\[f_2(e_1) = 1 - \alpha < 0 \Rightarrow 1 - \beta = f_3(e_1), \quad f_3(e_2) = 1 - \alpha < 0 \Rightarrow 1 - \beta = f_1(e_2),
\]

\[f_1(e_3) = 1 - \alpha < 0 \Rightarrow 1 - \beta = f_2(e_3).
\]

By Theorem 2.3, $\partial \Sigma : e_1 \rightarrow e_3 \rightarrow e_2 \rightarrow e_1$ is a heteroclinic cycle which is globally attracting when $\alpha > 2$. Figure 2 shows the heteroclinic cycle in the phase portrait.
2.2. **Heteroclinic cycles in 3-species Lotka-Volterra systems.** In this section we consider the special case of Lotka-Volterra systems. Now the conditions (11) and (12) applied to (2) with $N = 3$ become
\[
\begin{align*}
a_{12} &< 1 < a_{32}, \quad a_{23} < 1 < a_{13}, \quad a_{31} < 1 < a_{21}; \\
a_{12} &> 1 > a_{32}, \quad a_{23} > 1 > a_{13}, \quad a_{31} > 1 > a_{21}.
\end{align*}
\]
The first line of the above inequalities becomes the second line after the permutation $(1, 2, 3) \rightarrow (3, 2, 1)$. With
\[
a_{12} = \alpha_2, a_{13} = \beta_3, a_{23} = \beta_1, a_{25} = \alpha_3, a_{31} = \alpha_1, a_{32} = \beta_2, a_{11} = a_{22} = a_{33} = 1,
\]
(2) with $N = 3$ can be written as
\[
\begin{align*}
x_1' &= r_1 x_1(1 - x_1 - \alpha_2 x_2 - \beta_3 x_3), \\
x_2' &= r_2 x_2(1 - \beta_1 x_1 - x_2 - \alpha_3 x_3), \\
x_3' &= r_3 x_3(1 - \alpha_1 x_1 - \beta_2 x_2 - x_3).
\end{align*}
\]

Then the second line of the above inequalities for (2) with $N = 3$ becomes $\beta_i < 1 < \alpha_i$ for all $i \in I_3$ for (14). Under this condition and $\beta_i > 0$, it is shown [5, Lemma 2.1] that (14) has a unique interior fixed point $p$. Thus a direct application of Theorem 2.3 to (14) gives the following:

**Corollary 2.** Assume that (14) satisfies $0 < \beta_i < 1 < \alpha_i$ for all $i \in I_3$. If $p$ is globally asymptotically stable (hyperbolic with one-dimensional stable manifold $W^s(p)$ and globally repelling on $\Sigma$), then $\partial \Sigma$ is a globally repelling (attracting) heteroclinic limit cycle.

**Remark 1.** If we use the split Lyapunov method given in [25, 29, 3] to determine whether the interior fixed point $p$ is globally asymptotically stable or globally repelling on $\Sigma$, it is guaranteed by [29, Lemma 4.2] that $p$ is hyperbolic.

An obvious question arises: Does the condition $\prod_{i=1}^{3} (\alpha_i - 1) < \prod_{i=1}^{3} (1 - \beta_i)$ ($\prod_{i=1}^{3} (\alpha_i - 1) > \prod_{i=1}^{3} (1 - \beta_i)$) for local repulsion (attraction) of the heteroclinic cycle $\partial \Sigma$ ensure that this local behaviour extends globally? The answer is yes when $r_1 = r_2 = r_3$ [5], but unfortunately situation is much more complex when the components of $r$ are not all equal. Indeed, it is hinted in [5], and shown in [26], that for systems in Zeeman’s class 27 bifurcations may occur.

We present the following result to clarify this. Let
\[
A_0 = \begin{pmatrix}
1 & \alpha_2 & \beta_3 \\
\beta_1 & 1 & \alpha_3 \\
\alpha_1 & \beta_2 & 1
\end{pmatrix},
\]
\[
\Delta = r_1 r_2 p_1 p_2 (1 - \alpha_2 \beta_1) + r_1 r_3 p_1 p_3 (1 - \alpha_1 \beta_3) + r_2 r_3 p_2 p_3 (1 - \alpha_3 \beta_2),
\]
and, for any vector $v \in \mathbb{R}^N$, let $D(v) = \text{diag} [v_1, \ldots, v_N]$, the diagonal matrix.

**Theorem 2.4.** Assume that $0 < \beta_i < 1 < \alpha_i$ for all $i \in I_3$.

(a) If $\alpha_{j+1} \beta_j > 1$ ($4 = 1 \pmod{3}$) for some $j \in I_3$, then there is an open set $S_0 \subset \text{int} \mathbb{R}_+^3$ such that
\[
\forall r \in S_0, \quad (r_1 p_1 + r_2 p_2 + r_3 p_3) \Delta < r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0).
\]

For each $r \in S_0$, the fixed point $p$ is hyperbolic with a one-dimensional stable manifold and repels (locally) on $\Sigma$. 

(b) If \( \alpha_{j+1} \beta_j < 1 \) for some \( j \in I_3 \), then there is an open set \( S_1 \subset \text{int}\mathbb{R}_+^3 \) such that

\[
\forall r \in S_1, \ r_1 p_1 + r_2 p_2 + r_3 p_3 \Delta > r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0).
\]  
(17)

For each \( r \in S_1 \), the fixed point \( p \) is (locally) asymptotically stable.

(c) If (16) holds and \( \prod_{i=1}^3(\alpha_i - 1) < \prod_{i=1}^3(1 - \beta_i) \), then \( p \) is a local repeller on \( \Sigma \) and the heteroclinic cycle \( \partial \Sigma \) is locally repelling. Moreover, for each \( x^0 \in \text{int}\mathbb{R}_+^3 \setminus W^s(p) \), \( \omega(x^0) \) is a periodic orbit and, if \( x(t, x^0) \) is bounded for \( t < 0 \), \( \alpha(x^0) \) is either \( \{0\} \) or \( \{p\} \) or \( \partial \Sigma \) or a periodic orbit.

(d) If (17) holds and \( \prod_{i=1}^3(\alpha_i - 1) > \prod_{i=1}^3(1 - \beta_i) \), then the interior fixed point \( p \) is locally asymptotically stable and the heteroclinic cycle is locally attracting. Hence, for each \( x^0 \in \text{int}\Sigma \setminus \{p\} \), \( \alpha(x^0) \) is a periodic orbit and \( \omega(x^0) \) is either \( \{p\} \) or \( \partial \Sigma \) or a periodic orbit.

(e) If \( \prod_{i=1}^3(\alpha_i - 1) = \prod_{i=1}^3(1 - \beta_i) \), then (14) with \( s = s(1, \frac{\alpha_i - 1}{\omega_i(j)}, \frac{1 - \beta_i}{1 - \gamma_i}) \) for each \( s > 0 \) has a flat carrying simplex \( \Sigma = \{x \in \mathbb{R}_+^3 : x_1 + x_2 + x_3 = 1\} \), which is filled with periodic orbits between \( p \) and \( \partial \Sigma \).

(f) If there is a matrix \( D_0 = \text{diag}[d_1, d_2, d_3] \) with \( d_i > 0 \) for \( i \in I_3 \) such that \( D_0 A_0 + A_0^T D_0 \) is positive definite, then for all \( r \in \text{int}\mathbb{R}_+^3 \), the fixed point \( p \) is globally asymptotically stable and \( \partial \Sigma \) is a globally repelling heteroclinic limit cycle.

**Proof.** The Jacobian of (14) is \( J = \frac{\partial f}{\partial x}(p) = -D(r)D(p)A_0 \). So the characteristic polynomial of \( J \) is

\[
\det(D(r)D(p)A_0 + \lambda I) = \lambda^3 + \delta \lambda^2 + \Delta \lambda + r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0),
\]

where \( \Delta \) is given by (15) and \( \delta = r_1 p_1 + r_2 p_2 + r_3 p_3 \). From [5] we know that \( \det(A_0) > 0 \). Since \( r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0) \) and \( \delta \) are always positive, by Routh-Hurwitz criterion, all the eigenvalues of \( J \) have a negative real part if and only if the inequality in (17) holds. Clearly, \( \det(A_0) > 0 \) ensures that \( J \) has at least one negative eigenvalue \( -\lambda_1 < 0 \). If \( J \) has a pair of pure imaginary eigenvalues \( \pm i \mu \), then

\[
(\lambda + \lambda_1)(\lambda + i \mu)(\lambda - i \mu) = \lambda^3 + \lambda_1 \lambda^2 + \mu^2 \lambda + \lambda_1 \mu^2.
\]

This shows that there are a pair of pure imaginary eigenvalues if and only if

\[
(r_1 p_1 + r_2 p_2 + r_3 p_3) \Delta = r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0).
\]

Therefore, there are two eigenvalues with a positive real part if and only if the inequality in (16) holds.

(a) By \( \alpha_{j+1} \beta_j > 1 \), there is a negative term \( r_{j+1} r_j^2 p_{j+1} p_j^2(1 - \alpha_{j+1} \beta_j) \) in \( \delta \Delta \) of order \( O(r_j^2) \) but \( r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0) \) is linear in \( r_j \). Thus, when \( r_j \) is large enough and the other two components of \( r \) are fixed, \( r \) satisfies the inequality in (16). Then the existence of the open set \( S_0 \) follows from the strict inequality and continuity.

(b) The proof is similar to (a). By \( \alpha_{j+1} \beta_j < 1 \), there is a positive term \( r_{j+1} r_j^2 p_{j+1} p_j^2(1 - \alpha_{j+1} \beta_j) \) in \( \delta \Delta \) of order \( O(r_j^2) \) but \( r_1 r_2 r_3 p_1 p_2 p_3 \det(A_0) \) is linear in \( r_j \). Thus, when \( r_j \) is large enough and the other two components of \( r \) are fixed, \( r \) satisfies the inequality in (17). Then the existence of the open set \( S_1 \) follows from the strict inequality and continuity.

(c) From (a) we know that \( p \) is a local repeller on \( \Sigma \). From section 1 we also know that \( \partial \Sigma \) is a local repeller. Then, for each \( x^0 \in \text{int}\mathbb{R}_+^3 \setminus W^s(p) \), \( \omega(x^0) \) is a nonempty compact and connected subset of \( \text{int}\Sigma \setminus \{p\} \). So by the Poincaré-Bendixson theorem for three dimensional cooperative and competitive systems [18, Theorem 2.2] \( \omega(x^0) \) is a closed orbit. Clearly, if \( x(t, x^0) \) is bounded for \( t < 0 \), then either \( x^0 \) is in the
basin of repulsion of the origin, in this case \( \alpha(x^0) = \{0\} \), or \( x^0 \in \text{int}\Sigma \setminus \{p\} \). In the latter case, if \( x^0 \) is in the basin of repulsion of \( \{p\} \) then \( \alpha(x^0) = \{p\} \); if \( x^0 \) is in the basin of repulsion of \( \partial\Sigma \) then \( \alpha(x^0) = \partial\Sigma \); otherwise \( \alpha(x^0) \) is a closed orbit.

(d) From (b) we know that \( p \) is locally asymptotically stable and, from section 1, \( \partial\Sigma \) is locally attracting. Then, for each \( x^0 \in \text{int}\Sigma \setminus \{p\} \), \( \alpha(x^0) \) is a nonempty compact and connected subset of \( \text{int}\Sigma \setminus \{p\} \) so it is a closed orbit [20]. If \( x^0 \) is in the region of attraction of \( p \) then \( \omega(x^0) = \{p\} \); if \( x^0 \) is in the region of attraction of \( \partial\Sigma \) then \( \omega(x^0) = \partial\Sigma \); otherwise \( \omega(x^0) \) is a closed orbit.

(e) From [24] we know that the carrying simplex is flat if its edges are line segments. The condition for this is

\[
\begin{align*}
& r_1(1-\alpha_2) + r_2(1-\beta_1) = 0, \\
& r_1(1-\beta_3) + r_3(1-\alpha_1) = 0, \\
& r_2(1-\alpha_3) + r_3(1-\beta_2) = 0. \\
\end{align*}
\] (18)

When \( \prod_{i=1}^{3}(\alpha_i - 1) = \prod_{i=1}^{3}(1-\beta_i) \), \( r = s(1, \frac{2s-1}{\pi}, \frac{1-\beta_3}{\alpha_1-1})^T \) is a solution of (18) for any \( s \in \mathbb{R} \). Finally it is shown in [25, Theorem 4.8] that if the carrying simplex is flat then it is filled with periodic orbits.

(f) For any given \( r \in \text{int}\mathbb{R}_+^3 \), let \( D = D_0D(r)^{-1} \). Then the matrix \( DD(r)A_0 + A_0^TD(r)D = D_0A_0 + A_0^TD_0 \) is positive definite. By [21, Theorem 3.2.1], the fixed point \( p \) is globally asymptotically stable. By Corollary 2, \( \partial\Sigma \) is a globally repelling heteroclinic limit cycle.

\[ \square \]

2.3. Example 2. Consider system (14) with

\[
A_0 = \begin{pmatrix} 1 & 1.2 & 0.5 \\ 0.6 & 1 & 1.3 \\ 1.1 & 0.5 & 1 \end{pmatrix}. \] (19)

Since the leading principal minors of the symmetric matrix \( A_0 + A_0^T \) have positive values 2, 0.76, 0.288 respectively, \( A_0 + A_0^T \) is positive definite. By Theorem 2.4 (f), the interior fixed point \( p \) is globally asymptotically stable and \( \partial\Sigma \) is a globally repelling heteroclinic limit cycle (see Figure 3).

3. Heteroclinic limit cycles in \( N \)-dimensional systems with vanishing components. In this section, we consider system (1) with the assumption that, after relabeling the components if necessary, the three dimensional system (6) is the limiting subsystem of (1) as \( t \to +\infty \) (\( t \to -\infty \)), namely,

\[
\forall x^0 \in \text{int}\mathbb{R}_+^N, \ \forall i \in I_N \setminus I_3, \ \lim_{t \to +\infty} x_i(t, x^0) = 0, \ \left( \lim_{t \to -\infty} x_i(t, x^0) = 0 \right). \] (20)

Suppose the 3-dimensional Kolmogorov system (6) has a globally attracting (repelling) heteroclinic limit cycle \( \Gamma_0 \). Is this \( \Gamma_0 \) also a globally attracting (repelling) heteroclinic limit cycle of (1)? The aim of this section is to provide an answer to this question, which depends on the conditions to guarantee (20).

Lemma 3.1. Assume that the strongly competitive Kolmogorov system (1) satisfies the following conditions:

(i) There is a nonempty \( J_0 \subset I_N \) such that, if \( J_0 \neq I_N \),

\[
\forall k \in I_N \setminus J_0, \ \forall x^0 \in \text{int}\mathbb{R}_+^N(x^0 \in \text{int}\Sigma), \ x_k(x^0, t) = o(1) \] (21)

as \( t \to +\infty \) (\( t \to -\infty \)).
Figure 3. Globally repelling heteroclinic cycle in the competitive Lotka-Volterra system (14) with $A = A_0$ given by (19) and $r_1 = r_2 = r_3 = 1$. The globally repelling heteroclinic cycle is $e_1 \to e_2 \to e_3 \to e_1$.

(ii) there exist $i \in J_0$, $J_1 \subset J_0 \setminus \{i\}$ and $q \in \mathbb{R}_+^N$ with $q_j > 0$ if and only if $j \in J_1$ such that

$$\forall x \in S_0, \quad \sum_{j \in J_1} q_j f_j(x) > f_i(x) (f_i(x)), \quad (22)$$

where $S_0$ is the set of $x \in \mathbb{R}_+^N$ such that $f_j(x) \leq 0 \leq f_k(x)$ for some $j, k \in J_0$ and, if $J_0 \neq I_N$, $x \in \pi_\ell$ for all $\ell \in I_N \setminus J_0$.

Then there is a $\delta_0 > 0$ such that

$$x_\ell(t, x^0) = o(e^{-\delta_0|t|}) \text{ as } t \to +\infty \text{ (} t \to -\infty \text{)} \quad (23)$$

for all $x^0 \in \text{int}\mathbb{R}_+^N \ (x^0 \in \text{int}\Sigma)$ or $x^0 \in \bigcap_{j \in I_N \setminus J_0} \pi_j \ (x^0 \in \Sigma \cap \bigcap_{j \in I_N \setminus J_0} \pi_j)$ with $x^0_\ell > 0$ for all $\ell \in J_1 \cup \{i\}$.

Proof. For each $x^0 \in \mathbb{R}_+^N$, if $f_k(x^0) > 0$ for all $k \in I_N$ then $x^0$ is in the basin of repulsion of the origin so $\lim_{t \to -\infty} x(t, x^0) = 0$; if $f_k(x^0) < 0$ for all $k \in I_N$ then $x^0$ is in the basin of repulsion of $\infty$ so $x(t, x^0)$ is unbounded on $(-\infty, 0)$. Therefore, the carrying simplex $\Sigma$ must satisfy $\Sigma \subset S_0$ if $J_0 = I_N$ or $\Sigma \cap \bigcap_{\ell \in I_N \setminus J_0} \pi_\ell \subset S_0$ if $J_0 \neq I_N$. Then, for each $x^0 \in \text{int}\mathbb{R}_+^N \ (x^0 \in \text{int}\Sigma)$, since $\omega(x^0) \subset \Sigma \ (\alpha(x^0) \subset \Sigma)$, by condition (i) we have $\omega(x^0) \subset S_0 \ (\alpha(x^0) \subset S_0)$. If $x^0 \in \bigcap_{\ell \in I_N \setminus J_0} \pi_j \ (x^0 \in \Sigma \cap \bigcap_{j \in I_N \setminus J_0} \pi_j)$ with $x^0_\ell > 0$ for all $\ell \in J_1 \cup \{i\}$, as $\bigcap_{j \in I_N \setminus J_0} \pi_j$ is invariant, we also have $\omega(x^0) \subset S_0 \ (\alpha(x^0) \subset S_0)$. Let

$$\delta_1 = \min_{x \in S_0} [q^T f(x) - f_i(x)] \quad \left(\delta_1 = \max_{x \in S_0} [q^T f(x) - f_i(x)]\right).$$

Note that $S_0$ is compact. Then, by condition (ii), we have $\delta_1 > 0 \ (< 0)$. Thus, by the compactness of $S_0$, there is an $\varepsilon_0 > 0$ such that

$$\forall x \in B(S_0, \varepsilon_0) \cap \mathbb{R}_+^N, \quad q^T f(x) - f_i(x) \geq 2\delta_1 / 3 \leq 2\delta_1 / 3,$$
where $B(S_0, \varepsilon_0)$ is the $\varepsilon_0$ neighbourhood of $S_0$. Since $\omega(x^0) \subset S_0$ ($\alpha(x^0) \subset S_0$), there is a $T > 0$ such that $x(t, x^0) \in B(S_0, \varepsilon_0) \cap \mathbb{R}_+^N$ for $t \geq T$ ($t \leq -T$). Let

$$V(t) = x_i(t, x^0) \prod_{j \in J_1} (x_j(t, x^0))^{-\delta_j}.$$ 

Then, for $t \geq T$,

$$V'(t) = V(t) [f_i(x(t, x^0)) - q^T f(x(t, x^0))] \leq -\frac{2}{3} \delta_1 V(t)$$

for $t \leq -T$ so $V(t) \leq V(T)e^{-2\delta_1 (t-T)/3}$ for $t \geq T$ ($V(t) \leq V(-T)e^{-2\delta_1 (t+T)/3}$ for $t \leq -T$). Then, by the definition of $V(t)$ and the boundedness of $x(t, x^0)$ on $[0, +\infty)$ $((-\infty, 0])$, we obtain (23) with $\delta_0 = |\delta_1|/2$. □

**Remark 2.** In some cases, the inequalities (22) can be guaranteed by some inequalities on coefficients of the system. This will be demonstrated in Example 3 later in this section. For the particular class of Lotka-Volterra systems, there are various criteria available for some components of all solutions to vanish (see [22]–[28] for example). The proof of Lemma 3.1 is actually a generalisation of that of [28, Theorem 2], which is included in the corollary below for the case of $t \rightarrow +\infty$.

**Corollary 3.** Assume that the Lotka-Volterra system (2) satisfies the following conditions:

(i) There exists a nonempty $J_0 \subset I_N$ such that, if $J_0 \neq I_N$,

$$\forall k \in I_N \setminus J_0, \forall x^0 \in \text{int} \mathbb{R}_+^N (x^0 \in \text{int} \Sigma), \ x_k(t, x^0) = o(1)$$

as $t \rightarrow +\infty$ ($t \rightarrow -\infty$).

(ii) There exist $i \in J_0$, a nonempty $J_1 \subset J_0 \setminus \{i\}$ and $c \in \mathbb{R}_+^N$ with $c_j > 0$ if and only if $j \in J_1$ and $c_1 + \cdots + c_N = 1$ such that

$$\forall k \in J_0, \sum_{j \in J_1} c_j a_{jk} < a_{ik} (> a_{ik}).$$

(24)

Then there is a $\delta_0 > 0$ such that the solution of (2) satisfies

$$x_i(t, x^0) = o(e^{-\delta_0 |t|})$$

as $t \rightarrow +\infty$ ($t \rightarrow -\infty$)

for all $x^0 \in \text{int} \mathbb{R}_+^N (x^0 \in \text{int} \Sigma)$ or $x^0 \in \bigcap_{j \in I_N \setminus J_0} \pi_j$ ($x^0 \in \Sigma \cap (\bigcap_{j \in I_N \setminus J_0} \pi_j)$) with $x^0_\ell > 0$ for all $\ell \in J_1 \cup \{i\}$.

**Proof.** Take $q_j = r_i c_j/r_j$. Then (24) implies

$$\sum_{j \in J_1} q_j f_j(x) = r_i \left(1 - \sum_{k \in I_N \setminus J_0} \sum_{j \in J_1} c_j a_{jk} x_k\right) > f_i(x) (> f_i(x))$$

for all $x \in \mathbb{R}_+^N$ with $x_\ell = 0$ if $\ell \in I_N \setminus J_0$. Then condition (ii) of Lemma 3.1 is satisfied and the conclusion follows. □

**Theorem 3.2.** Suppose (1) meets the following conditions:

(a) The property (20) is obtained by $N - 3$ successive applications of Lemma 3.1 to (1).

(b) As a subsystem of (1), (6) meets the requirement of Theorem 2.3 so it has a globally repelling (attracting) interior fixed point $p$ and a globally attracting (repelling) heteroclinic limit cycle $\Gamma_0$.

(c) Each of the three axial fixed points $e_1, e_2, e_3$ of (6) is a hyperbolic fixed point of (1).
Then the following conclusions hold:

(i) As a fixed point of (1), p has an (N − 2)-dimensional stable manifold \(W^s(p) \subset \text{int}\mathbb{R}_+^N\) and \((N − 3)\)-dimensional unstable manifold \(W^u(p) \subset \text{int}\Sigma\), i.e.

\[
\forall x^0 \in W^s(p), \lim_{t \to +\infty} x(t, x^0) = p \quad (\forall x^0 \in W^u(p), \lim_{t \to -\infty} x(t, x^0) = p).
\]

(ii) For each \(i \in I_3\), there is a \(k\) satisfying \(2 \leq k \leq N - 1\) such that \(e_i\) as a fixed point of (1) has a \(k\)-dimensional stable manifold \(W^s(e_i) \subset \partial\mathbb{R}_+^N\) and an \((N - k)\)-dimensional unstable manifold \(W^u(e_i) \subset \partial\mathbb{R}_+^N\).

(iii) The heteroclinic cycle \(\Gamma_0\) of (6) is also a globally attracting (repelling) heteroclinic limit cycle of (1), i.e.

\[
\forall x^0 \in \text{int}\mathbb{R}_+^N \setminus W^s(p), \omega(x^0) = \Gamma_0 \quad (\forall x^0 \in \text{int}\Sigma \setminus \{p\} \cup W^u(p)), \quad \alpha(x^0) = \Gamma_0).
\]

Proof. (i) With \(p = (p_1, p_2, p_3, 0, \ldots, 0)^T \in \mathbb{R}_+^N\) as a fixed point of (1), we first show that \(\{p\}\) is strictly above (below) \(\gamma_j\) for all \(j \in I_N \setminus I_3\). By the assumption (a), in the last of the \((N - 3)\) applications of Lemma 3.1, we have \(J_0 = I_3 \cup \{i\}, J_1 \subset I_3\). Since \(\{p\} = \gamma_1 \cap \gamma_2 \cap \gamma_3 \cap (\bigcap_{j=4}^N \pi_j)\), we have \(f_k(p) = 0\) for \(k \in I_3\). By Lemma 3.1 condition (ii), \(0 = \sum_{j \in J} q_j f_j(p) > f_i(p) (< f_i(p))\). Thus, \(\{p\}\) is strictly above (below) \(\gamma_i\). Repeating the above process, we see that \(\{p\}\) is strictly above (below) \(\gamma_j\) for all \(j \in I_N \setminus I_3\). Then, by assumption (b), the Jacobian \(J(p)\) has exactly \(N - 2\) negative eigenvalues \((N - 3\) positive eigenvalues) so the conclusion (i) follows.

(ii) This follows directly from assumptions (b) and (c).

(iii) By condition (a) and conclusion (i), for each \(x^0 \in \text{int}\mathbb{R}_+^N \setminus W^s(p)\) \((x^0 \in \text{int}\Sigma \setminus W^u(p))\), we have \(\omega(x^0) \subset \Gamma_0 \quad (\alpha(x^0) \subset \Gamma_0)\). If \(\omega(x^0) \neq \Gamma_0 \quad (\alpha(x^0) \neq \Gamma_0)\), then following the same proof of Theorem 2.3 we would have obtained a contradiction. Therefore, we must have \(\omega(x^0) = \Gamma_0 \quad (\alpha(x^0) = \Gamma_0)\).

Corollary 4. For Lotka-Volterra system (2), the statement of Theorem 3.2 is still valid after the replacements of (1) by (2), Lemma 3.1 by Corollary 3, Theorem 2.3 by Corollary 2 and (6) by (14).

3.1 Example 3. Consider system (1) with

\[
\begin{align*}
f_1(x) &= 1 - x_1 - \beta \frac{x_2}{1 + x_3} - \alpha x_3 - \sum_{j=4}^N a_{1j} g_j(x_j), \\
f_2(x) &= 1 - \alpha x_1 - x_2 - \beta \frac{x_3}{1 + x_1} - \sum_{j=4}^N a_{2j} g_j(x_j), \\
f_3(x) &= 1 - \beta \frac{x_1}{1 + x_2} - \alpha x_2 - x_3 - \sum_{j=4}^N a_{3j} g_j(x_j), \\
f_4(x) &= 1 - a_{11} x_1 - a_{12} x_2 - a_{13} x_3 - \sum_{j=4}^N a_{4j} g_j(x_j),
\end{align*}
\]

where \(N > 3, i \in I_N \setminus I_3, \alpha > 2, 0 < \beta < 1\), the \(a_{ij}\) are positive constants, and each \(g_k \in C^1(\mathbb{R}_+, \mathbb{R}_+)\) satisfies

\[
g'_k(s) > 0 \quad \forall s \in \mathbb{R}_+, \quad g_k(0) = 0 \quad \text{and} \quad g_k(s) \to +\infty \quad \text{as} \quad s \to +\infty.
\]
Then (1) with (25) is strongly competitive. Assume that the positive coefficients 
\( a_{ij} \) satisfy, \( \forall i \in I_N \setminus I_3, \forall j \in I_i \setminus I_3 \) and \( \forall k \in I_3, \)
\[
a_{ik} > \frac{1}{3}(1 + \alpha + \beta), \quad a_{ij} > \frac{1}{3}(a_{1j} + a_{2j} + a_{3j}).
\] (26)

Then \((f_1(x) + f_2(x) + f_3(x))/3 > f_N(x)\) for all \( x \in \mathbb{R}_+^N \setminus \{0\} \). By Lemma 3.1, every solution of (1) with (25) in \( \text{int} \mathbb{R}_+^N \) satisfies \( x_N(t) = o(e^{-\delta_0t}) \) \((t \to +\infty)\) for some \( \delta_0 > 0 \). If \( N > 4 \), then (26) for \( i = N - 1 \) ensures that \((f_1(x) + f_2(x) + f_3(x))/3 > f_{N-1}(x)\) for all \( x \in (\mathbb{R}_+^N \cap \pi_N) \setminus \{0\} \). By Lemma 3.1 again, every solution of (1) with (25) in \( \text{int} \mathbb{R}_+^N \) satisfies \( x_{N-1}(t) = o(e^{-\delta_0t}) \) \((t \to +\infty)\) for some \( \delta_0 > 0 \). Repeating the above process \( N - 3 \) times, we obtain \( \lim_{t \to +\infty} x_j(t, x^0) = 0 \) for all \( x^0 \in \text{int} \mathbb{R}_+^N \) and \( j \in I_N \setminus I_3 \). From Example 1 given in the previous section we know that system (13) as a subsystem of (25) has a globally repelling interior fixed point \( p \) and a globally attracting heteroclinic limit cycle \( \Gamma_0 \) by the application of Theorem 2.3. From \( \alpha > 2, 0 < \beta < 1 \) and (26) we have \( a_{ik} > 1 \), so \( f_i(e_k) < 0 \), for all \( 1 \leq k \leq 3 < i \leq N \). This means that the Jacobian of (1) with (25) at each \( e_k \) has \( N - 1 \) negative eigenvalues and 1 positive eigenvalue. As the conditions (a)–(c) of Theorem 3.2 are all met, the fixed point \( p \) has an \((N - 2)\)-dimensional stable manifold \( W^s(p) \) and \( \omega(x^0) = \Gamma_0 \) for all \( x^0 \in \text{int} \mathbb{R}_+^N \setminus W^s(p) \).

### 3.2. Example 4.
Consider the competitive Lotka-Volterra system

\[
\begin{align*}
  x_1' &= x_1(1 - x_1 - 6x_2 - \frac{2}{3}x_3 - 4x_4), \\
  x_2' &= x_2(1 - \frac{2}{3}x_1 - x_2 - 6x_3 - 4x_4), \\
  x_3' &= x_3(1 - 6x_1 - \frac{2}{3}x_2 - x_3 - 4x_4), \\
  x_4' &= x_4(1 - 4x_1 - 4x_2 - 4x_3 - 6x_4).
\end{align*}
\] (27)

The steady states are

\[(0,0,0,0)^T, e_1 = (1,0,0,0)^T, e_2 = (0,1,0,0)^T, e_3 = (0,0,1,0)^T, (0,0,0,1/6)^T\]

and \( p = (\frac{3}{25}, \frac{3}{25}, \frac{3}{25}, 0)^T \). There is no interior fixed point so all omega limit sets are contained in the boundary. The conclusion for this example can be achieved by analysing \( S' = (x_1 + x_2 + x_3)' \) and \( x_4' \) together. However, it is more convenient to use Corollary 3 and Theorem 3.2. Note that for \( c \in \mathbb{R}_+^4 \) with \( c_1 = c_2 = c_3 = \frac{1}{4} \) and \( c_4 = 0 \), we have \( c_1 + \cdots + c_4 = 1 \) and \( \sum_{j=1}^3 c_j a_{jk} = \frac{2a_3}{9} < 4 = a_{4k} \) for \( k \in I_3 \), \( \sum_{j=1}^3 c_j a_{jk} = 4 < 6 = a_{44} \). Then, by Corollary 3, the 4th component of every solution of (27) in \( \text{int} \mathbb{R}_+^4 \) vanishes as \( t \to +\infty \). Comparing the subsystem of (27) with (3), we see that the May-Leonard subsystem of (27) on \( \pi_4 \) has \( \alpha = 6 \) and \( \beta = 2/3 \). Since \( \alpha + \beta = 20/3 > 2 \), the interior fixed point \( \frac{2}{3} (1,1,1)^T \) is globally repelling and there is a globally attracting heteroclinic cycle \( \Gamma_0 \); \( e_1 \to e_2 \to e_3 \to e_4 \) (see Figure 4 for the phase portrait in the case \( \alpha = 5/2, \beta = 1/2 \)). Clearly, each \( e_j \) is a hyperbolic fixed point. By Theorem 3.2, \( p \) has a 2-dimensional stable manifold \( W^s(p) \) in \( \text{int} \mathbb{R}_+^4 \) and \( \omega(x^0) = \Gamma_0 \) for all \( x^0 \in \text{int} \mathbb{R}_+^4 \setminus W^s(p) \).
Figure 4. Dynamics of (27) projected on the 3-dimensional probability simplex. Parameter values: $\alpha = 5/2, \beta = 1/2$. The globally attracting heteroclinic cycle is $E1 \rightarrow E2 \rightarrow E3 \rightarrow E1$.

3.3. Example 5. This example demonstrates the application of Corollary 4 for a globally repelling heteroclinic cycle. Consider system (2) with $N = 4$ and

$$A = \begin{pmatrix} 1 & 1.2 & 0.5 & 2 \\ 0.6 & 1 & 1.3 & 1 \\ 1.1 & 0.5 & 1 & 2 \\ 0.8 & 0.5 & 0.7 & 1 \end{pmatrix}.$$  \hspace{1cm} (28)

The average of the first three rows of $A$ is $(A_1 + A_2 + A_3)/3 = (0.9, 0.9, 2.8/3, 5/3)$. Since each of its components is greater than that of $A_4$, by Corollary 3 every solution of (2) with (28) in $\text{int}\mathbb{R}^4_+$ satisfies $\lim_{t \to -\infty} x_4(t) = 0$. From example 2 we know that (14) with (19) as a subsystem of (2) with (28) has a globally attracting interior fixed point $p$ and a globally repelling heteroclinic limit cycle $\Gamma_0$. From (28) and the Jacobian matrix we can see that each of the axial fixed points $e_1, e_2, e_3$ is hyperbolic. Then, by Corollary 4, $p$ has a one-dimensional unstable manifold $W^u(p) \subset \Sigma$ and $\Gamma_0$ is a globally repelling heteroclinic limit cycle such that $\alpha(x^0) = \Gamma_0$ for all $x^0 \in \text{int}\Sigma \setminus W^u(p)$.

4. Heteroclinic limit cycles in $N$-dimensional competitive Kolmogorov systems. In this section, we consider system (1) with a globally attracting (repelling) fixed point $p \in \text{int}\mathbb{R}^N_+$. Let $E$ be the set of all fixed points of (1) in $\mathbb{R}^N_+$. For any $y \in \mathbb{R}^N_+$ and $\varepsilon > 0$, we denote the open ball centered at $y$ with radius $\varepsilon$ by $B(y, \varepsilon)$, the closed ball by $\overline{B}(y, \varepsilon)$ and the sphere by $\partial B(y, \varepsilon) = \overline{B}(y, \varepsilon) \setminus B(y, \varepsilon)$.

We now define a concept which will be used in the condition of the next theorem.
Definition 4.1. A set $\Gamma$ is called an extended trajectory if $\Gamma$ satisfies the following two requirements:

(i) $\Gamma$ consists of a finite number of heteroclinic trajectories $\Gamma_1, \ldots, \Gamma_k$ and $k + 1$ fixed points $p_0, p_1, \ldots, p_k$.

(ii) Each $\Gamma_i$ connects $p_{i-1}$ to $p_i$ with flow direction from $p_{i-1}$ to $p_i$ so that the flow direction on $\Gamma$ is from $p_0$ to $p_k$.

An extended trajectory $\Gamma$ is called simple if $\Gamma$ does not contain any cycle, i.e. $p_0, \ldots, p_k$ are distinct.

Theorem 4.2. Assume that every fixed point of (1) is hyperbolic and the following conditions hold.

(i) There is a fixed point $p \in \text{int}\mathbb{R}_+^N$ that is a global repeller (attractor) on $\text{int}\Sigma$.

(ii) $f_i(e_i) > 0$ (modulo $N$) ($< 0$) for $i \in I_N$ and $f_i(e_i) < 0$ ($> 0$) otherwise.

(iii) The set $E$ of fixed points is finite and if $E' = E \setminus \{0, p, e_1, \ldots, e_N\} \neq \emptyset$ then each fixed point $q \in E'$ restricted to $\Sigma$ is a saddle point.

(iv) For each $i \in I_N$, restricted to $\Sigma \cap \pi_{i+1}$, $e_i$ is an attractor (a repeller). Moreover, every nontrivial trajectory in $\Sigma \cap \pi_{i+1}$ is heteroclinic and every extended trajectory is simple. Furthermore, for each fixed point $q \in (\Sigma \cap \pi_{i+1}) \setminus \{e_i\}$, $W^s(q) \subset \pi_{i+1}$ ($W^s(q) \subset \pi_{i+1}$).

Then $\omega(x_0) = \Gamma^0$ ($\alpha(x_0) = \Gamma^0$) for every $x_0 \in \text{int}\mathbb{R}_+^N \setminus U$ ($x_0 \in \text{int}\Sigma \setminus U$), where $\Gamma^0$ is the heteroclinic cycle joining the $N$ axial fixed points $e_1 \to e_2 \to \cdots \to e_N \to e_1$ ($e_1 \to e_N \to \cdots \to e_2 \to e_1$) and $U$ is the union of $E$ and the stable (unstable) manifolds of all saddle points.

Remark 3. Note that condition (ii) ensures that every axial fixed point is a saddle point with an $(N - 1)$-dimensional stable manifold and a one-dimensional unstable manifold (2D stable and $(N - 2)$D unstable). Suppose $p$ is a global repeller on $\Sigma$. Then, by assumptions (ii) and (iii), every nonzero fixed point is a saddle point so its stable manifold is of dimension less than $N$. Thus, $U$ is the union of a finite number of stable manifolds of dimension less than $N$ and the set of fixed points. Since $\omega(x_0) = \Gamma^0$ holds for all $x_0 \in \text{int}\mathbb{R}_+^N \setminus U$, by Definition 1.1 $\Gamma^0$ is a globally attracting heteroclinic limit cycle. However, if $p$ is a global attractor, by (ii)–(iv) every fixed point $q \in (E \cap \pi_{i+1}) \setminus \{0, e_i\}$ is a saddle point with $W^s(q) \subset \pi_{i+1}$ of dimension at least 2 and $W^u(q)$ of dimension at most $N - 2$. Therefore, $U$ is the union of a finite number of unstable manifolds of dimension less than $N - 1$ and the set of fixed points. Since $\alpha(x_0) = \Gamma^0$ holds for all $x_0 \in \text{int}\mathbb{R}_+^N \setminus U$, by Definition 1.1 again $\Gamma^0$ is a globally repelling heteroclinic limit cycle.

Proof of Theorem 4.2. Suppose $p$ is a global repeller on $\Sigma$. Then, for each $x_0 \in \text{int}\mathbb{R}_+^N \setminus U$, we have $\omega(x_0) \subset \partial\Sigma$. Thus, $\omega(x_0) \cap \partial\Sigma \cap \pi_{i+1} \neq \emptyset$ for some $i$ ($0 \leq i \leq N - 1$).

We first show that $e_i \in \omega(x_0)$. For any $x_1 \in \omega(x_0) \cap \partial\Sigma \cap \pi_{i+1}$, if $x_1 = e_i$ then $e_i \in \omega(x_0)$. Suppose $x_1 \neq e_i$ and $x_1$ is a fixed point. Since $W^u(x_1) \subset \pi_{i+1}$ by (ii) and (iv) and $\omega(x_0) \cap (W^u(x_1) \setminus \{x_1\}) \neq \emptyset$ by Lemma 2.2, there is an extended trajectory $\tilde{\Gamma}_1$ from $x_1$ to a fixed point $x_2$ satisfying $\tilde{\Gamma}_1 \subset \omega(x_0) \cap \partial\Sigma \cap \pi_{i+1}$. If $x_2 = e_i$ then $e_i \in \omega(x_0)$. Otherwise $\tilde{\Gamma}_1$ can be further extended from $x_1$ through $x_2$ to a fixed point $x_3$ such that $\tilde{\Gamma}_1 \subset \omega(x_0) \cap \partial\Sigma \cap \pi_{i+1}$. Since $E$ is finite by (iii) and every extended trajectory is simple, repeating the above process a finite number of times we obtain $e_i \in \omega(x_0)$. If $x_1$ is not a fixed point then there is an extended trajectory $\tilde{\Gamma}_1$ through $x_1$ from a fixed point $y_1 \in \omega(x_0) \cap \partial\Sigma \cap \pi_{i+1}$ to
another fixed point \( x_2 \in \omega(x_0) \cap \partial \Sigma \cap \pi_{i+1} \). Then the same reasoning used above also shows that \( e_i \in \omega(x_0) \).

Since \( e_i \) is a saddle point and its unstable manifold \( W^u(e_i) \) in \( \mathbb{R}^N \) consists of \( e_i \) and the heteroclinic trajectory \( \Gamma_i \) from \( e_i \) to \( e_{i+1} \) in the 2-dimensional \( x_i x_{i+1} \)-plane, by Lemma 2.2, \( \Gamma_i \subset \omega(x_0) \). Thus, we also have \( e_{i+1} \in \omega(x_0) \). Repeating the above process \( N \) times, we obtain \( \Gamma_0 \subset \omega(x_0) \).

Note that in \( \mathbb{R}^N \), \( \Gamma_i \) is the only trajectory going out from \( e_i \). This shows that \( \omega(x_0) = \Gamma_0 \).

If \( p \) is a global attractor, for each \( x^0 \in \text{int} \Sigma \setminus U \), replacing \( \omega(x^0) \) by \( \alpha(x^0), W^s(x_1) \) by \( W^s(x_1), W^u(e_i) \) by \( W^u(e_i) \), the above reasoning is still valid. \( \square \)

4.1. Example 6. Consider the following 4-dimensional May-Leonard system

\[
x'_i = x_i (1 - A_i x), \quad i \in I_4
\]

with interaction matrix

\[
A = \begin{pmatrix}
1 & \beta & \gamma & \alpha \\
\alpha & 1 & \beta & \gamma \\
\gamma & \alpha & 1 & \beta \\
\beta & \gamma & \alpha & 1
\end{pmatrix},
\]

where \( \alpha, \beta \) and \( \gamma \) are positive constants. Then \( p \in \text{int} \mathbb{R}^4_+ \) with \( p_i = \frac{1}{1+\alpha+\beta+\gamma} \) for \( i \in I_4 \) is the unique interior fixed point. Let

\[
\Omega(A) = \left( \frac{\beta - 1}{1 - \alpha} \right)^4.
\]

Then, if \( \gamma > 1 \) and

\[
0 < \alpha < 1, \beta > 1, \alpha + \beta > 2
\]

so that \( \Omega(A) > 1 \), we know from [1, 7] that the heteroclinic cycle \( \Gamma_0 : e_1 \to e_2 \to \cdots e_4 \to e_1 \) is at least locally attracting. The question is whether \( \Gamma_0 \) is globally attracting.

In what follows we assume that either (32) or (33) holds, where

\[
\alpha < 1 < \beta, \quad 2 < \gamma + 1 < \alpha + \beta \quad (\leq \alpha^2 + \beta^2),
\]

\[
\alpha > 1 > \beta, \quad 2 > \gamma + 1 > \alpha + \beta > \alpha^2 + \beta^2.
\]

(In (32), the condition \( \alpha^2 + \beta^2 > \alpha + \beta \) holds automatically since \( \alpha + \beta > 2 \).) In particular, by (31), (32) ensures that \( \Gamma_0 \) is at least locally attracting. We wish to use Theorem 4.2 to show that \( \Gamma_0 \) is globally attracting.

First we show that \( p \) is a global repeller (attractor) if (32) ((33)) holds. We use the split Lyapunov method, and in particular Theorem 6.7 in [25] (see also [29]). The Jacobian matrix of (29) at \( p \) is

\[
-D(p)A = -\text{diag}[p_1, p_2, p_3, p_4]A = -\frac{1}{1+\alpha+\beta+\gamma} \begin{pmatrix}
1 & \beta & \gamma & \alpha \\
\alpha & 1 & \beta & \gamma \\
\gamma & \alpha & 1 & \beta \\
\beta & \gamma & \alpha & 1
\end{pmatrix}.
\]

The matrix \( D(p)A \) has an eigenvalue \( \lambda_0 = 1 \) with a corresponding left eigenvector \( v^T = (1, 1, 1, 1) \). Since \( D(v) \) is the identity matrix, we have

\[
\frac{1}{2}(AD(v)^{-1} + (AD(v)^{-1})^T) = \frac{1}{2}(A + A^T) = \\
\begin{pmatrix}
1 & \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} \\
\frac{\alpha+\beta}{2} & 1 & \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} \\
\frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & 1 & \frac{\alpha+\beta}{2} \\
\frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & 1
\end{pmatrix}.
\]
Then the symmetric matrix
\[
M = \begin{pmatrix}
2 - (\alpha + \beta) & 1 - \gamma & 1 + \gamma - (\alpha + \beta) \\
1 - \gamma & 2(1 - \gamma) & 1 - \gamma \\
1 + \gamma - (\alpha + \beta) & 1 - \gamma & 2 - (\alpha + \beta)
\end{pmatrix}
\]
is obtained from \(\frac{1}{2}(A + A^T)\) by subtracting the last column from every column, subtracting the last row from every row, and then deleting the last row and column. The leading principal minors of \(M\) are
\[
2 - (\alpha + \beta), (\gamma - 1)[2(\alpha + \beta) - \gamma - 3], -4(\gamma - 1)^2(\alpha + \beta - \gamma - 1).
\]
From these minors we find that \(M\) is negative definite provided \(2 < \gamma + 1 < \alpha + \beta\), and \(M\) is positive definite if \(\alpha + \beta < \gamma + 1 < 2\). Hence \(M\) is negative (positive) definite under the assumptions of \((32)\) \((33)\). By \([25, \text{Theorem 6.7}]\) \(p\) is a global repellor (attractor) and moreover by \([29, \text{Theorem 4.4}]\) the global attractor is globally asymptotically stable. From Remark 1 we know that \(p\) is hyperbolic.

Thus, the assumption \((32)\) or \((33)\) ensures the conditions (i) and (ii) of Theorem 4.2 for \((29)\). Note that condition (ii) of Theorem 4.2 implies that each axial fixed point is hyperbolic.

Next we examine the existence of non-axial boundary fixed points. Let the vector \(p^* = (x_1^*, x_2^*, x_3^*)^T \in \mathbb{R}^3\) be defined by
\[
p^* = \begin{pmatrix} 1 & \beta & \gamma \\ \alpha & 1 & \beta \\ \gamma & \alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\delta} \begin{pmatrix} (1 - \alpha)(\beta - \gamma) + (1 - \beta)^2 \\ (\gamma - 1)(\alpha + \beta - \gamma - 1) \\ (1 - \alpha)^2 + (\beta - 1)(\gamma - \alpha) \end{pmatrix}, \tag{34}
\]
where \(\delta = 1 + \gamma(\alpha^2 + \beta^2) - \gamma^2 - 2\alpha\beta\). Now we have
\[
\delta = (1 - \gamma^2) + (\gamma - 1)(\alpha^2 + \beta^2) + \alpha^2 + \beta^2 - 2\alpha\beta
\]
\[
= (\gamma - 1)(-1 + \gamma) + (\alpha^2 + \beta^2) + (\alpha - \beta)^2.
\]
Hence if \((32)\) holds we have \(\gamma > 1\) and \(\alpha + \beta > \gamma + 1\), so that \(\alpha^2 + \beta^2 > \alpha + \beta\) and so \(\delta > 0\). On the other hand if \((33)\) holds then \(\gamma < 1\) and \(\alpha^2 + \beta^2 < \alpha + \beta\) so
\[
\delta > (\gamma - 1)(-1 + \gamma) + (\alpha + \beta) + (\alpha - \beta)^2 > (\alpha - \beta)^2 > 0.
\]
Moreover, if either \((32)\) or \((33)\) hold we have \(x_2^* = \frac{1}{\delta}(\gamma - 1)(\alpha + \beta - \gamma - 1) > 0\).

Now we must show that for either \((32)\) or \((33)\) we have \(x_1^*(\alpha, \beta) = \frac{1}{\delta}((1 - \alpha)(\beta - \gamma) + (1 - \beta)^2) > 0\) and \(x_3^*(\alpha, \beta) = \frac{1}{\delta}((1 - \beta)(\alpha - \gamma) + (1 - \alpha)^2) > 0\). If \((32)\) holds then \((1 - \alpha) > 0\) and \(\beta - \gamma > 1 - \alpha > 0\) so that \(x_1^*(\alpha, \beta) > 0\), and since \(\beta - 1 > \gamma - \alpha > 0\) we also have \(x_3^*(\alpha, \beta) > 0\). For the case \((33)\), \(1 - \alpha < 0\) and \(\beta - \gamma < 1 - \alpha < 0\) so that \(x_1^*(\alpha, \beta) > 0\), and also \(\beta - 1 < 0\), \(\gamma - \alpha < \gamma - 1 < 0\) so that \(x_3^*(\alpha, \beta) > 0\). The conclusion is that whenever \((32)\) or \((33)\) is satisfied, \(q_4 = (x_1^*, x_2^*, x_3^*, 0)^T\) is a boundary fixed point. Similarly we find that
\[
Q_1 = \left(\frac{1}{1 + \gamma}, 0, \frac{1}{1 + \gamma}, 0\right)^T, \quad Q_2 = \left(0, \frac{1}{1 + \gamma}, 0, \frac{1}{1 + \gamma}\right)^T,
\]
\[
q_1 = (0, x_1^*, x_2^*, x_3^*)^T, \quad q_2 = (x_3^*, 0, x_1^*, x_2^*)^T, \quad q_3 = (x_2^*, x_3^*, 0, x_1^*)^T
\]
and \(q_4\) are the only fixed points in \(E \setminus \{0, p, e_1, e_2, e_3, e_4\}\).

We next check that these boundary fixed points satisfy the conditions (iii) and (iv) of Theorem 4.2 under the assumption \((33)\). Note that \(p^* \in \text{int}\mathbb{R}_+^4\) defined by
(34) is a fixed point of the subsystem of (29) with \( x_4 = 0 \). The symmetric matrix

\[
S = \begin{pmatrix}
1 & \beta & \gamma \\
\alpha & 1 & \beta \\
\gamma & \alpha & 1
\end{pmatrix} + \begin{pmatrix}
1 & \beta & \gamma \\
\alpha & 1 & \beta \\
\gamma & \alpha & 1
\end{pmatrix}^T = \begin{pmatrix}
2 & \alpha + \beta & 2\gamma \\
\alpha + \beta & 2 & \alpha + \beta \\
2\gamma & \alpha + \beta & 2
\end{pmatrix}
\]

has leading principal minors 2, 4 - (\( \alpha + \beta \))^2 and 4(1 - \( \gamma \))(2 + 2\( \gamma \) - (\( \alpha + \beta \))^2). Since \( \gamma < 1 \), \( \alpha + \beta < 1 + \gamma < 2 \) so that 4 - (\( \alpha + \beta \))^2 > 0. Moreover, (\( \alpha + \beta \))^2 < (1 + \( \gamma \))^2 < 2(1 + \( \gamma \)), so that all leading principal minors are positive and \( S \) is positive definite. By [21, Theorem 3.2.1], \( p^* \) is globally asymptotically stable in \( \text{int}\mathbb{R}^3 \). Now identifying \( p^* \) with \( q_4 \), we show that \( J_4 \), the Jacobian of (29) at \( q_4 \), has last row \((0, 0, 0, 1 - \beta x_1^* - \gamma x_2^* - \alpha x_3^*) \) and so has an eigenvalue \( \lambda_1 = 1 - (\beta, \gamma, \alpha, 1)q_4 = 1 - (\beta, \gamma, \alpha, 1)(x_1^*, x_2^*, x_3^*, 0)^T \). The set \( \gamma_1 \cap \gamma_2 \cap \pi_4 \) is the line segment \( Q_1z \) where \( z = (\frac{\alpha - \beta}{\alpha - \beta^*}, \frac{1 - \gamma}{\alpha - \beta^*}, 0, 0)^T \) and \( q_4 \in Q_1z \). As \( (\beta, \gamma, \alpha, 1)Q_1 = \frac{\alpha + \beta}{1 + \gamma} < 1 \) and, by \( \beta < \gamma < 1 < \alpha \),

\[
(\beta, \gamma, \alpha, 1)z = \frac{\beta(\alpha - \beta) + \gamma(1 - \gamma)}{\alpha - \beta\gamma} = \frac{\alpha(\beta - 1) + \beta(\gamma - \beta) + \gamma(1 - \gamma)}{\alpha - \beta\gamma} + 1 < \frac{\gamma(1 - \beta) + \alpha(\beta - 1)}{\alpha - \beta\gamma} + 1 < 1,
\]

we also have \((\beta, \gamma, \alpha, 1)q_4 < 1\) which shows \( \lambda_1 > 0 \). The remaining eigenvalues of \( J_4 \) are those of the 3 \( \times \) 3 submatrix \( J_4 \) obtained by removing the last row and column from \( J_4 \). The 3-dimensional competitive May-Leonard system obtained by restricting to \( \pi_4 \) is globally stable, as shown above by using that \( S \) is positive definite when (33) holds. By global stability, the matrix \(-A\) is \( D\)-stable, i.e. \(-AD\) is stable for any positive diagonal matrix \( D \). Taking \( D = D(x_1^*, x_2^*, x_3^*) \) we see that \( J_4 \) is a \( D\)-stable matrix, which shows that \( J_4 \) has one positive and 3 negative eigenvalues so that \( q_4 \) is a saddle point with a 3-dimensional stable manifold in \( \pi_4 \) and a one-dimensional unstable manifold. Thus, restricted to \( \Sigma \), \( q_4 \) is a saddle point. Similarly, \( q_1 \), \( q_2 \), and \( q_3 \) are all saddle points restricted to \( \Sigma \).

Next, the subsystem of (29) with \( x_2 = x_4 = 0 \) and \( 0 < \gamma < 1 \) is a typical planar competitive Lotka-Volterra system: it can be shown by phase plane analysis that \((\frac{1}{1 + \gamma}, \frac{1}{1 + \gamma})^T\) is a globally asymptotically stable node. Since \((\alpha, 1, \beta, \gamma)Q_1 = (\beta, \gamma, \alpha, 1)Q_1 = \frac{\alpha + \beta}{1 + \gamma} < 1\), \( J_1 \), the Jacobian of (29) at \( Q_1 \), has two positive eigenvalues \( 1 - \frac{\alpha + \beta}{1 + \gamma} \). As \( J_1 \) has spectrum \( \{-1, \frac{\gamma - 1}{\gamma + 1}, \frac{\alpha - \beta + \gamma + 1}{\gamma + 1}, \frac{-\alpha - \beta + \gamma + 1}{\gamma + 1}\} \), so that when (33) holds \( J_1 \) has 2 negative, 2 positive eigenvalues. Therefore, \( Q_1 \) is a saddle point with a 2-dimensional stable manifold in \( \pi_2 \cap \pi_4 \) and a 2-dimensional unstable manifold. Thus, restricted to \( \Sigma \), \( Q_1 \) is a saddle point. The same reasoning applied to \( Q_2 \) shows that the 2-dimensional stable manifold \( S(Q_2) \) is in \( \pi_1 \cap \pi_3 \) and \( Q_2 \) restricted to \( \Sigma \) is also a saddle point. Then condition (iii) of Theorem 4.2 is met.

The phase portraits on \( \Sigma \cap \pi_j \), \( j \in I_4 \), are given in Figure 5. These show that condition (iv) of Theorem 4.2 is satisfied. By Theorem 4.2, when (33) holds the heteroclinic cycle \( \Gamma_0 \) is a global repeller in the sense of Definition 1.1. Figure 6 shows the heteroclinic limit cycle in the phase portrait for the case \( \beta = 9/4, \alpha = 1/2, \gamma = 3/2 \).
Figure 5. The phase portraits on $\Sigma \cap \pi_j$ $(1 \leq j \leq 4)$.

Figure 6. Dynamics of (29) with interaction matrix given by (30) projected on the 3-dimensional probability simplex. Parameter values: $\beta = 9/4, \alpha = 1/2, \gamma = 3/2$. The globally attracting heteroclinic cycle is $E_1 \to E_2 \to E_3 \to E_4 \to E_1$. 
5. Conclusion. For strongly competitive autonomous Kolmogorov systems, we have explored the global dynamics when a heteroclinic limit cycle is a global attractor (repeller). Starting from three-dimensional systems, we have obtained necessary as well as sufficient conditions for the existence of a globally attracting (repelling) heteroclinic limit cycle. In particular, for three dimensional Lotka-Volterra systems, we have shown that a locally attracting (repelling) heteroclinic limit cycle may not necessarily be globally attracting (repelling). Then, for an \( N \)-dimensional system with a three-dimensional subsystem as its limit system, i.e. the same \( N-3 \) components of all solutions in \( \text{int}\mathbb{R}^N_+ \) vanish as \( t \to +\infty \) (\( t \to -\infty \)), we have shown that a globally attracting (repelling) heteroclinic limit cycle of the subsystem can also be globally attracting (repelling) in \( \text{int}\mathbb{R}^N_+ \) under certain conditions. For general \( N \)-dimensional systems, we have provided a sufficient condition for the existence of a globally attracting (repelling) heteroclinic limit cycle. We have also demonstrated the applications of our results by various examples.

There remain many outstanding problems of interest concerning stability of heteroclinic cycles. Even for the class of strongly competitive autonomous Kolmogorov systems, to the best of our knowledge, not much is known about the global dynamics characterised by a globally attracting (repelling) heteroclinic limit cycle. We conclude this paper with a few problems for future investigation.

5.1. Open Problem 1. In section 1 we mentioned that there are some deeper results, based largely on Poincaré maps, on heteroclinic cycles for general dynamical systems with symmetry (see, for example, [9, 10]) but we have not investigated to what extent they can be applied to our system (1). Thus open problem 1 is to investigate these results in the context of ecological models and use Lyapunov function techniques rather than Poincaré maps. It would also be interesting to investigate the subclass of systems (1) with symmetry.

5.2. Open Problem 2. The heteroclinic cycles in our results consist of only axial fixed points and heteroclinic trajectories joining them. The 2nd open problem is to investigate the possibility of globally attracting (repelling) heteroclinic limit cycles that consist of fixed points not necessarily all axial and heteroclinic trajectories. Find conditions for existence of such limit cycles.

5.3. Open Problem 3. For general system (1), Theorem 4.2 is only a sufficient condition for the existence and global attraction (repulsion) of a heteroclinic limit cycle. Open problem 3 is to explore alternative and weaker conditions.

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