The nonlinear mechanics of slender structures undergoing large deformations

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The deformation of slender structures (i.e., structures that are much longer in one direction than in the other two directions) may be described by elastic rod theory if one is interested in phenomena on length scales much larger than the lateral dimensions of the structure. A simplified version of this theory, beam theory, suffices if only small deformations need to be considered. Many traditional structural engineering applications are adequately modelled by a beam: bridges and buildings are simply not designed to suffer large deformations. But in other applications of structural mechanics deformations may be large, meaning that nonlinear geometric effects have to be taken into account. In these cases full (nonlinear) rod theory is the theory of choice. Besides engineering applications this approach is also increasingly being applied to molecular biology, for instance in the study of DNA packing and supercoiling. Aided by the improved experimental techniques for manipulating structures at the micro- and nanoscale, there is great activity in the application of mechanical models to biological filaments.

Mathematically, a rod is modelled as a curve in space, i.e., a one-dimensional object, with effective mechanical properties such as bending and torsional stiffnesses [1]. Although the problems considered are often statics problems, mathematically they have a strong relation with problems in dynamics: arclength along the rod plays a role similar to that of time in a dynamical system (ordinary differential equation). In its simplest form this analogy (so called Kirchhoff's Dynamic Analogy [19]) is seen between the deformations of an elastic strut (or Euler elastica) and the motions of a swinging pendulum. A similar, albeit less well known, analogy exists between a twisted rod and a spinning top, and precession of the top then corresponds to a helical deformation of the rod. Thus the extensive field of rigid-body dynamics can be brought to bear on the deformation of slender structures. Important tools in this research are therefore modern analytical as well as numerical techniques from nonlinear dynamics. This includes techniques for dealing with chaotic dynamics, which in rods takes the form of 'spatial chaos' [12]. For a sufficiently long intrinsically straight rod the energetically favourable deformation is a localised one in which the structure tends to the trivial, flat, state towards the ends of the rod. This state is described by a homoclinic solution of the dynamical system, i.e., a solution that connects a saddle point (describing the straight rod) to itself [13, 10]. Such homoclinic solutions are characteristic of many chaotic systems and have been the focus of intensive research in recent years (e.g., [3, 4]).

Engineering problems where rod theory is applied include the looping of ocean cables [7] and the buckling of oilwell dring strings [26]. Drill strings nowadays can be over 5 km long; they are confined to narrow (possibly curved) boreholes, and therefore present the problem of structural deformation and buckling in the presence of a constraining surface. Transitions in the buckling shape of the drill string, for instance from a snaky to a helical pattern, are known to occur but still poorly understood. For a twisted rod in permanent contact with a cylinder critical loads have been found at which the rod collapses into a helical shape [10].

The same model for a rod on a cylinder describes the supercoiling of a DNA molecule into a so-called plectoneme, a plied structure in which two strands wind around each other. The problem of this ply is one of a rod winding on the outside (rather than the inside) of a cylinder, namely the cylinder, of radius equal to the radius of the rod, on which the strand centrelines wind [6, 24]. There is also a relation with chromatin fibres in which DNA coils around protein (histone) cores in the first stage of the DNA compaction process [21].

There is great interest in the application of elastic rod theory to the supercoiling and packing of DNA molecules (see [22] for a review) and proteins such as collagen. Owing to its double helical nature, DNA is unusually stiff (in both bending and torsion) for a polymer, and hence well described by a rod. During cell divisions, DNA is packaged into a small volume where elastic forces compete with electrostatic forces. This packaging has to happen in an orderly fashion so that the process can be undone in a reliable way for the next cycle of DNA replication to take place. A similar packaging process occurs in bacteriophages. A bacteriophage is a virus that infects bacteria. It typically consists of a head (or capsid) with several tails. The capsid acts as a vesicle containing tightly packed genetic material (either closed or open strands of DNA) that, once the virus is in contact with a bacterial cell, is injected into the cell, where it gets transcribed by the bacterial biosynthetic machinery [15, 18]. Bacteriophages can be viewed as model systems for animal cell viruses, and much research effort is being put into getting a better understanding of this packaging of DNA (e.g., [20, 17]).

Models of plied structures have also been used to get a better understanding of the stability of twisted textile yarns [9, 25]. Yarn spun from staple fibres such as wool or cotton is held together by twist. To stably wind the spun yarn on bobbins additional twist needs to be inserted. To twist a yarn it is necessary to subject it to an axial torque. A model of the process must therefore include the bending and torsional stiffness of the yarn, i.e., describe the yarn as a rod, not a string [8]. The production speed of most yarn spinning processes is limited by the rate at which twist can be inserted. If the balance between feed-through speed, spindle speed and applied tension is not right, then twist-induced instabilities (snarls) occur, which may lead to undesirable imperfections in the final fabric or to yarn breakage and a costly waste of time.

Rods with intrinsic curvature [5] are good models for helical filaments, which in biology occur for instance in the form of metastable cholesterol ribbons in the process of gallstone formation in gallbladder bile [28]. These helices occur in two distinctive forms: a high and a low pitch state. In [23] a tension-induced switching between two stable helical states of different pitch was observed experimentally. Rod theory predicts that this stretching instability only occurs in helices whose radius to pitch ratio is sufficiently large and whose torsional stiffness exceeds its bending stiffness [16]. It also predicts a further instability under torsional loading which gives rise to what may be called twisted ribbons [11]. At other critical loads non-helical solutions are found to emerge. This latter critical behaviour is related to the chaotic nature of the underlying nonlinear equilibrium equations. A further consequence of this spatial chaos is the existence of a multitude of localised forms for helical springs familiar from telephone cords.

The study of strings and rods in a magnetic field is of interest to electrodynamic space tethers. A space tether is a long cable used to connect spacecraft to each other or to other orbiting bodies such as space stations, boosters, payload, etc. in order to transfer energy and momentum [2]. Electrodynamic tethers have been proposed in various forms and are generally believed to hold great potential for future space missions. One purpose of such tethers is to bring down space debris (derelict spacecraft, etc.), which is threatening future space flight. An example is the so-called Short Electrodynamic Tether (SET) proposed by the European Space Agency [27]. Electrodynamic tethers interact with the earth's magnetic field in order to create an electric current along the tether. This is made possible by electric devices called contactors, attached at the ends of the tether, which emit or absorb electrons in the surrounding space plasma, thereby closing the electric circuit. As a result of the current induced according to Faraday's Law an electrodynamic force is generated that can be used to thrust or drag the system without the need for chemical fuel. Unlike the majority of tethers, which can be treated as cables, the prototype SET is designed to withstand torsional and bending forces so that it can be spun in order to get an ideal relative orientation where the tether interacts efficiently with the earth's magnetic field. Therefore, the SET, and similar tethers, need to be modelled as spinning rods (not cables) [14].

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