

# Supplementary material

E.L. Starostin, G.H.M. van der Heijden

## 1. Möbius strip centrelines:

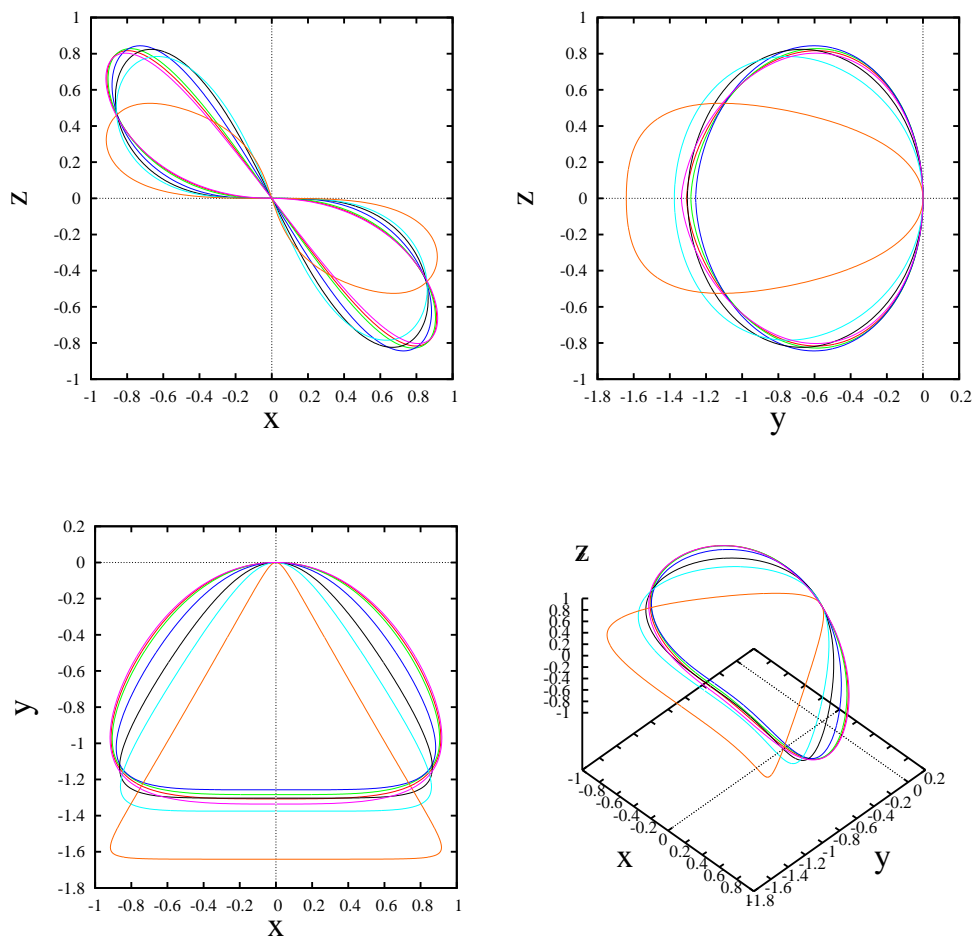


Figure 1: Projections and 3D shape of the centreline of Möbius strips for  $w = 0$  (magenta), 0.1 (red), 0.2 (green), 0.5 (blue), 0.8 (black), 1.0 (cyan) and 1.5 (orange).

2. Diagram of torsion against curvature of the strip's centreline:

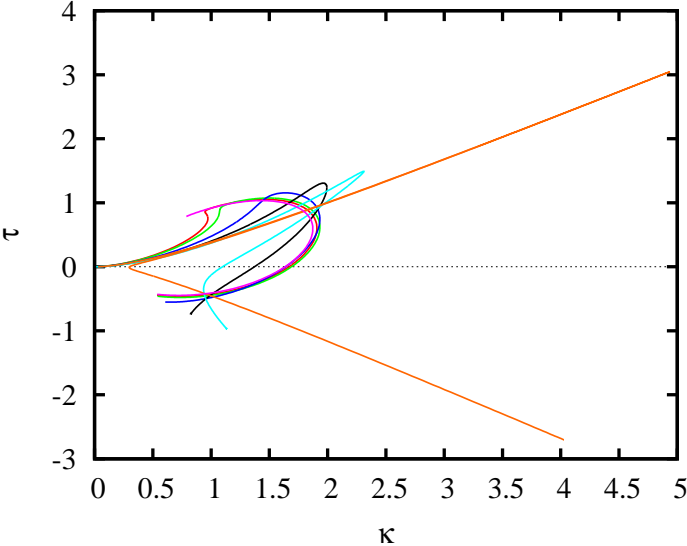


Figure 2: Colours as in Fig. 1.

### 3. Energy, twist and writhe of the strip's centreline:

The normalised energy  $U = V/(Dw)$  of the strip is shown in Fig. 3. The only meaningful energy estimate available in the literature, at the particular value of  $w = \pi/30$ , is that of Gravesen & Willatzen [2], who minimise the energy ((3) in the main text) for the parametrised family of developable Möbius strips (not in equilibrium) constructed in [4]. Their solution gives  $U = 9.22$ ; at the same width we find the much lower value  $U = 5.93$ .

Also shown in Fig. 3 is the twist  $Tw = \frac{1}{2\pi} \int_0^L \tau ds$  of the strip as a function of  $w$ .  $Tw$  may be written as  $Tw = Lk - Wr$ , where  $Lk$  is the linking number of the Möbius cord and  $Wr$  is the writhe of the centreline.  $Wr$  is commonly used as a measure for the spatial deformation of a 3D curve. It can be defined as the crossing number of the curve averaged over all viewing directions [1].

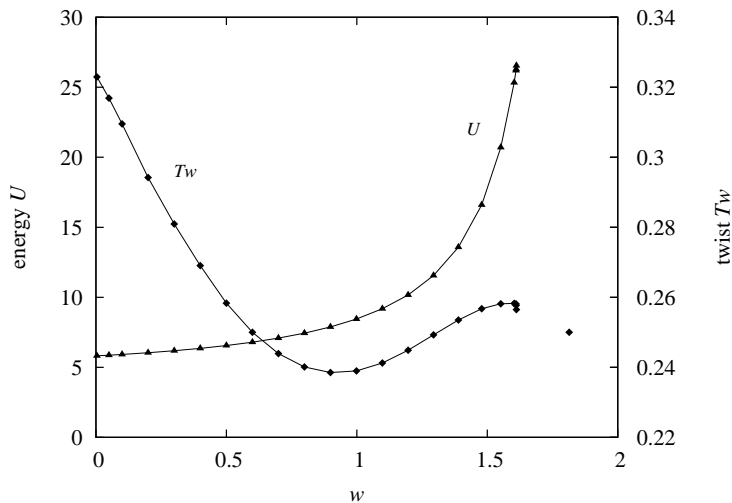


Figure 3: Energy  $U$  and twist  $Tw$  against  $w$ . The results suggest that the energy diverges as the critical value  $w_c = \pi/\sqrt{3} = 1.8138\dots$  is approached. In the same (flat) limit both  $Tw$  and  $Wr$  can be analytically shown to be  $1/4$ .

## 4. Previous work on equilibrium shapes:

A narrow elastic Möbius strip may be modelled as a rod with non-circular cross-section taken in the limit where one bending stiffness goes to infinity [3]. This corresponds to a strip formed by the binormal as it moves along the rod's centreline. Such a model, which is reasonable for highly extensible materials, cannot show flat triangular regions with bending energy localisation near their vertices. Clearly, a thin rod model cannot describe the dependence of the centreline on the width of the strip, which becomes important for sufficiently wide strips.

In the case of inextensible material, such as paper, a rod model fails to describe the developable surface of *any* Möbius strip, narrow or wide: it does not capture the essential geometrical properties of such a strip, for instance the fact that the torsion  $\tau$  is zero in the inflection point. Mahadevan & Keller [3] try to overcome this deficiency by artificially modifying the curvature and torsion functions over a small interval (they do not comment on how this can be done without violating the closedness of the centreline in 3D space). However, this modified solution has  $\eta = \tau/\kappa \neq 0$  (hence  $\beta \neq \pi/2$ ) at the inflection point, which prohibits the generators to line up at this point (where the Frenet frame flips). Therefore, the sought developable surface simply does not exist.

## 5. Shapes of a developable strip with $Lk = \frac{3}{2}$ :

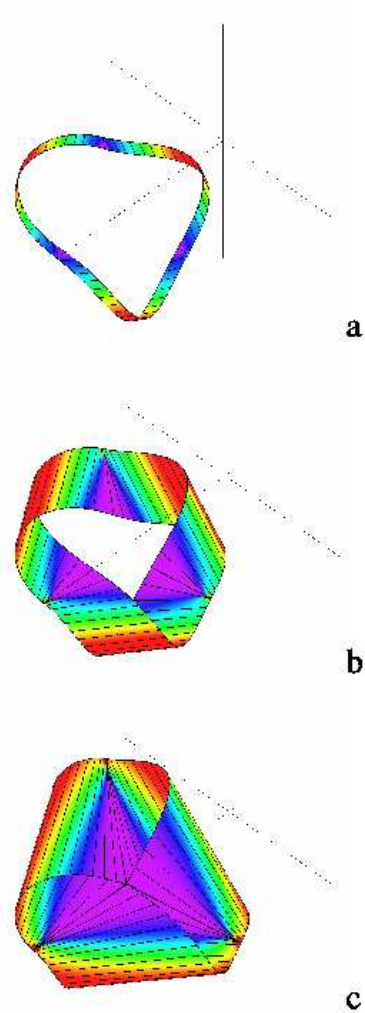


Figure 4: 3D shapes of a one-sided developable closed strip with  $Lk = \frac{3}{2}$  and  $w = 0.0573$  (a),  $0.336$  (b) and  $0.666$  (c). (The length of the strip is again fixed at  $2\pi$ .)

## References

- [1] F.B. Fuller, The writhing number of a space curve, *Proc. Nat. Acad. Sci. USA* **68**, 815–819 (1971).
- [2] J. Gravesen, M. Willatzen, Eigenstates of Möbius nanostructures including curvature effects, *Phys. Rev. A* **72**, 032108 (2005).
- [3] L. Mahadevan, J.B. Keller, The shape of a Möbius band, *Proc. R. Soc. Lond. A* **440**, 149–162 (1993).
- [4] T. Randrup, P. Røgen, Sides of the Möbius strip, *Arch. Math.* **66**, 511–521 (1996).