Review of:

Vibrational Mechanics: Nonlinear Dynamic Effects, General Approach, Applications

Iliya I. Blekhman (transl. Minna Perelman)

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Fast vibrations may have strong and sometimes unexpected effects on nonlinear mechanical systems. Familiar examples are the stabilisation of the inverted pendulum by rapid oscillation of its point of support, the shift of a vibrating compass needle and the self-synchronisation of unbalanced rotors resting on a common foundation. Vibrations play an important role in many areas of engineering and manufacturing. In some cases they have undesirable effects (such as in the self-unscrewing of nuts of vibrating machinery), but in a growing number of industrial applications the effects of fast vibration are exploited to aid dynamical processes and devices. Uses include the vibrational transport of material (based on the change in effective rheological characteristics such as dry friction and viscosity coefficients), the vibrational sinking of piles, vibrational cutting, the vibrational separation of granular mixtures, and the vibration of liquid or granular material in order to enhance chemical reaction.

The book by Blekhman gives a large collection of examples from a wide range of engineering applications and proposes a general mechanical approach (and philosophy) for the study of problems involving vibrations. It is a slightly expanded English edition of the original Russian version published in 1994. The author's mathematical analysis proceeds along the same lines in all the examples considered, with the mechanical system assumed to be of the form

$$m\ddot{x} = F(\dot{x}, x, t) + \Phi(\dot{x}, x, t, \omega t), \tag{1}$$

where x is an n-dimensional generalised co-ordinate, ω has to be thought of as the frequency of vibration, and the dot denotes differentiation with respect to slow time t. Solutions of this equation are sought of the form

$$x = X(t) + \Psi(t, \tau), \qquad \tau = \omega t, \tag{2}$$

satisfying

$$< \Psi(t, \tau) >:= \frac{1}{2\pi} \int_0^{2\pi} \Psi(t, \tau)\, \mathrm{d}\tau = 0 \tag{3}$$

and the coupled system of integro-differential equations

$$m\ddot{X} = F(\dot{X}, X, t) + < \tilde{F}(\dot{X}, X, \dot{\Psi}, \Psi, t) > + < \Phi(\dot{X} + \dot{\Psi}, X + \Psi, t, \tau) >, \tag{4}$$

$$\begin{aligned} m\ddot{\Psi} = {} & \tilde{F}(\dot{X}, X, \dot{\Psi}, \Psi, t) - < \tilde{F}(\dot{X}, X, \dot{\Psi}, \Psi, t) > + \Phi(\dot{X} + \dot{\Psi}, X + \Psi, t, \tau) \\ & - < \Phi(\dot{X} + \dot{\Psi}, X + \Psi, t, \tau) >, \end{aligned} \tag{5}$$

where

$$\tilde{F}(\dot{X}, X, \dot{\Psi}, \Psi, t) = F(\dot{X} + \dot{\Psi}, X + \Psi, t) - F(\dot{X}, X, t).$$

The first of these equations is the result of inserting (2) into (1), averaging both sides with respect to the fast time τ and using (3). The second equation is then obtained by subtracting (4) from (1).

Thus, if X and Ψ solve (4) and (5), then $x = X + \Psi$ solves the original equation (1). This technique is called the method of direct separation of motion.

Now, if the Ψ component is much faster than the slow component X, then we may consider equation (5) with X and $\dot{X}$ 'frozen', i.e., constant. Once a solution $\Psi = \Psi^*(\dot{X}, X, t, \tau)$ has then been obtained, (4) can be written as

$$m\ddot{X} = F(\dot{X}, X, t) + V(\dot{X}, X, t), \tag{6}$$

where

$$V(\dot{X}, X, t) =< \tilde{F}(\dot{X}, X, \dot{\Psi}^*, \Psi^*, t) > + < \Phi(\dot{X} + \dot{\Psi}^*, X + \Psi^*, t, \tau) > . \tag{7}$$

In practice one will often have to resort to approximate solution methods for the fast component Ψ^*, such as a sum of a small number of harmonics. If Ψ is considered to be small compared to X, then F and Φ may be linearised with respect to Ψ (and possibly $\dot{\Psi}$) to find a solution. Throughout the book it is assumed that the fast motion Ψ^* is asymptotically stable so that the potential V is well-defined over a certain range of initial conditions of the fast variables Ψ. If there are several stable fast motions then the potential V will depend on which motion is considered.

Equation (6) is the main equation of what the author calls *vibrational mechanics*. It is an equation for the slow dynamics with an effective potential due to the fast dynamics. Blekhman goes to great pains to interpret this situation in terms of two observers, an ordinary observer O who sees everything, including the fast motion, and a *vibrational* observer V who does not (or will not) see the fast motion and therefore attributes the potential V to additional slow forces or moments acting on the system, much like a non-inertial observer feels centrifugal and Coriolis forces. The forces corresponding to V are called vibrational forces. Since the Ψ motion is invisible to observer V, the Ψ variables are called hidden (or ignored) variables.

A simple example will do much to illustrate the author's viewpoint. Consider a pendulum whose point of suspension is oscillating in the vertical direction. The equation of motion is given by

$$I\ddot{\theta} = -mgl\sin\theta + ml\omega^2 A\sin\theta\cos\omega t, \tag{8}$$

where θ is the angle of deflection from the downward vertical, I is the moment of inertia, m the mass, l the distance from the centre of mass to the axis of the pendulum, g the acceleration due to gravity, and A and ω the amplitude and frequency of vibration. Writing

$$\theta = \alpha(t) + \psi(t, \tau),$$

and going through the steps outlined above (linearising and solving the fast equation) one obtains the slow equation

$$I\ddot{\alpha} + mgl\sin\alpha - V(\alpha) = 0, \qquad \text{where} \qquad V(\alpha) = -\frac{(mlA\omega)^2}{4I}\sin 2\alpha.$$

Equilibria of the slow motion (called *quasi-equilibria* of the pendulum) include the hanging ($\alpha = 0$) and inverted ($\alpha = \pi$) positions of the pendulum, the latter being stable provided that

$$\frac{ml(A\omega)^2}{2gI} > 1,$$

in agreement with the classical result on the stabilisation of the upside-down pendulum. For the mathematical pendulum, which has $I = ml^2$, this condition can be written as $m(A\omega)^2/2 > mgl$, saying that the kinetic energy of vibration must exceed the potential energy acquired by the pendulum in rising to a height l. Small oscillations about the inverted solution are described by

$$I\ddot{\alpha} + (mgl + (mlA\omega)^2/2I)\alpha = 0. \tag{9}$$

In the language of vibrational mechanics, observer V, who does not see the fast forcing, will explain the stable inverted position of the pendulum as the result of a spring support with a spring constant proportional to $(A\omega)^2$, as described by the last term on the left-hand side of (9). Note that (9) also tells us that a pendulum clock subjected to vertical vibration is always fast.

Incidentally, equation (9) also illustrates an important limitation of the vibrational mechanics approach since although the stability of the inverted position is reproduced correctly, the stability of the hanging position ($\alpha = 0$) is not. As is well known, the hanging solution of a parametrically forced pendulum is unstable inside so-called resonance tongues in the frequency-amplitude parameter plane. This result, which follows from a standard averaging argument, is not reproduced by vibrational mechanics, according to which the hanging solution is always stable (the assumptions implicit in the vibrational mechanics approach turn out to be valid only near the origin of the frequency-amplitude parameter plane).

The book under review is a monograph in five parts aimed at researchers. Its main objective is to discuss a great many vibrational systems, to identify the vibrational forces and to find (approximate) expressions for these forces. The author's approach is mainly formal. Chapters 3 and 4 of Part I give some results on the validity of the employed solution method, but these chapters do not form the best part of the book. In Chapter 3 the author shows that for certain solutions of a certain type of systems possessing a small parameter ($1/\omega$), the results agree with Bogolyubov's theory of averaging (although no statement of this theory is given). This is followed in Chapter 4 by a sketchy review of mathematical results from the Russian literature, but the lack of detail and explanation make this chapter virtually unreadable, useful only as a pointer to past work. Sentences such as "It was shown by Malakhova [111,361] that the sign of Valeyev and Ganiyev could also be obtained by the use of the theorem of Malkin [363]" (p. 87) and "It is also necessary to remark that the theorem of Beletsky and Kasatkin [57] is in good agreement with the results obtained by Hapayev and Shinkin [235,493]" (p. 88) make one feel quite an outsider to the story.

Mathematicians will recognise the method of direct separation of motion as defining the stage for inertial manifolds, including the dimensional reduction by rapid attraction towards a slow manifold resulting in the fast motion being slaved to the slow motion. The author does not mention this modern connection. It should be remembered though, that the book was originally written in the early 90s.

After the fundamentals of Part I we get a large number of applications in the remaining 17 chapters of Parts II to V. Part II deals with pendulum and rotor systems, starting in Chapter 5 with the inverted (multi-link) pendulum (with references to the recent Western literature on this topic). Also discussed in detail is the interesting and apparently still not fully understood behaviour of what is called the Chelomei pendulum, a parametrically driven rod (either rigid or elastic) with a washer free to slide along its length. The phenomenon of dynamic stabilisation of statically unstable systems such as compressed beams is also briefly mentioned.

Rotor systems are covered in Chapters 6, 7 and 8. Central here is the phenomenon of self-synchronisation, first reported and explained in 1665 by Huygens for two pendulum clocks hung from the same beam. It remains a remarkable phenomenon that unbalanced rotors that are not connected to each other either kinematically or electrically, but merely placed on a common flexible base, tend to synchronise their motion. This tendency may in fact be so strong that even if one of the rotors is switched off it keeps running in step with the ones still switched on. The effect is caused by vibrations of the common support, but these vibrations, as in the case of the Huygens pendula, may be hard to notice (Huygens initially held vibrations of the air responsible). In terms of vibrational mechanics, observer V will say that there are elastic shafts or springs connecting the rotors.

Next, in Part III, systems with dry friction are considered. Chapter 9 deals with vibrational transportation: a particle on a vibrated rough surface with an asymmetric Coulomb friction law drifts in the direction of least resistance. For observer V this involves the transformation of dry friction into viscous friction. Related to this are the vibrational sinking of piles, percussive drilling, as well as the separation (self-sorting) of granular mixtures by vibration of the container. All these technologies make use of the action of forces of alternating sign. Many more applications are discussed in the following two chapters.

Part IV, which starts with Chapter 12, deals with vibro-rheology and has a lot more to say on dry friction and fluids, granular media and suspensions under vibration. Chapter 18 is new in the English edition and discusses approaches to controlling the vibro-rheological properties of mechanical systems. This is quite a new area of research aimed at designing *dynamical materials*, materials with new dynamical properties induced by vibration.

Part V has two more chapters: Chapter 20 on particles moving in an oscillating non-uniform field (of which the parametric pendulum is an example) and Chapter 21 on resonance (synchronisation) in celestial mechanics. The bibliography that follows this final chapter contains 601 references, almost all to the Russian literature, which makes the book also useful as an extensive survey of that literature.

The material covered by the book is interesting and one is likely to come across a few surprising facts and observations. The author's wide angle and general treatment are highly original; I am not aware of any other book to compare this work with. It is a pity the material is so badly presented, particularly in Part I. The author's style is long-winded and repetitive, but perhaps his greatest mistake is to wait more than 100 pages before giving the first example. The whole theoretical discussion in Part I is done without going through the computational steps of one single example. An early example would have brought the material much more alive.

The text may not be well written, the translation makes it far worse. For instance, we read "ungenerated matrix" for "nonsingular matrix", forces are said to be "potential" instead of "conservative", and the book is full of adjective phrases such as "the averaged over the period Lagrangian" (p. 80) and "a stable according to the first approximation periodic solution" (p. 87), while a colleague with some knowledge of Russian explained to me that the frequently used "are answered by" means "corresponds to". Chapter 4 has a good dose of long meaningless phrases such as "... the expressions L_s, $L^{(I)}$ and $L^{(II)}$ are called respectively the eigenproper Lagrangians of the autosynchronized objects, and Lagrangians of the systems of the carrying and carried connections between the objects" (p. 80). To make things worse still, the book (clearly not copy edited) has a large number of typographical errors. One annoying result is that all the references to equations in Chapter 2 are off by one.

In conclusion, if you can put up with the author's 'relativistic' talk and convoluted sentences, then this book offers a wealth of interesting mechanical problems and phenomena, many of which could form the topic of further research, and I suspect that in the process of reading the text one learns a great deal of Russian too.