Topological invariants of knots: three routes to the Alexander Polynomial

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MT4000 Double Project
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May 14, 2005

O time! thou must untangle this, not I;
It is too hard a knot for me to untie!

William Shakespeare
Twelfth Night, Act II, Scene 2

Contents

Introduction

This project was originally entirely based around JW Alexander's 1927 Paper *Topological invariants of knots and links*, in which the author introduces the Alexander polynomial. While doing background reading on the subject, however, I became aware that calculation of the polynomial could be approached from three different viewpoints: combinatorially, as in Alexander's original formulation; geometrically, via constructions called *Seifert surfaces* and algebraically, by considering the *group* of the knot.

In considering these different viewpoints, I have increased the original scope of the project in order to show—pun intended—how knot theory ties together different areas of mathematics.

Because of the increased breadth of this project, I do not prove all assertions in detail, but attempts to sketch a proof are made where possible.

I also intend this project to be readable, in the most part, by someone with little mathematical experience. Because of this, there is extra explanation of mathematical concepts such as groups and topological surfaces; informal descriptions are used where possible and I have tried to include useful analogies along the way.

To show application of all the theories and to maintain a sense of continuity, all of the examples in this document feature two knots: 3_1 and 5_2. This is so that the reader becomes familiar with the knots and so the different mathematical viewpoints as mentioned above can be more easily compared.

Edward Long

Chapter 1

Knots, links and their invariants

1.1 History and knot basics

Knots are objects that we are all familiar with in everyday life and it comes as a surprise to some that there is a considerable amount of research devoted to their study in a mathematical context. The origins of knot theory are linked to physics; in the latter part of the 19th century a physical theory associated to Lord Kelvin proposed that the universe was filled with a substance known as *ether* and it was the way matter intertwined with this substance that brought about properties of the chemical elements. It was therefore believed by some that the study of knots would enlighten physicists as to the deepest mysteries of the universe. Because of this, there was a drive to tabulate and enumerate as many knots as possible and to be able to tell, especially in the case of more complex knots, whether two knots were the same, or indeed whether something was knotted at all or could be unravelled to what is referred to by knot theorists as the *unknot*. The Scottish physicist Peter Tait spent years compiling tables of knots in an attempt to produce what he believed could be a table of chemical elements defined through this theory.

The order in which knots are tabulated is by *crossing number*, which is the number of times the curve of the knot crosses itself when the knot is drawn in its simplest form. Of course, finding the simplest form of the knot is a difficult task in itself and many knots in Tait's table were later found to have simpler diagrams or to be repeats of other knots in the table. Tabulation of the knots also leads to the traditional notation for a knot in the form N_m, where N is the crossing number and the knot appears as the mth knot with that crossing number in the knot table. The knot table lists only what are known as the *prime knots*. Knots which are not prime are called *composite knots* and these are knots that can be decomposed by cutting through two strands of the 'string' and retying the ends to give two separate nontrivial knots. The *trefoil* or *clover leaf* (shown below) is the simplest nontrivial prime knot and is the simplest to tie. It is the only prime knot with crossing number 3 and is denoted 3_1. Two trefoils can be combined to form a *reef* *knot*, which is an example of a composite knot. There are two prime knots with crossing number 5 and below is shown the knot 5_2. These knots will feature in all of the examples in the rest of this document. A table of all prime knots of crossing number up to 7 is given at the back of this document.

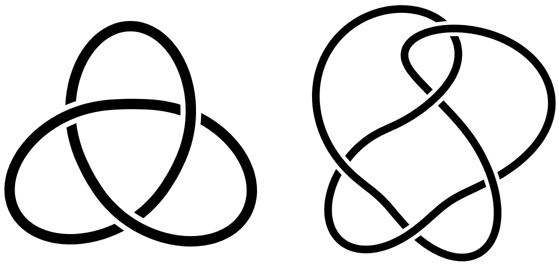


Figure 1.1: The knots 3_1 and 5_2

Unfortunately for Tait, the ethereal theory was discredited in 1897 by experimental evidence gathered at Case-Western Reserve University by Albert Abraham Michelson and Edward Morley. Added to this the advancements made in atomic theory (for example Ernest Rutherford's nuclear model of the atom and Joseph John Thomson's discovery of the electron), the physics community soon lost all interest in knots and study that followed was by pure mathematicians and amateur puzzle-solvers interested in properties of the knots themselves.

So what is a mathematical knot? In the real world we think of a knot as a length of string or rope wound around itself with the ends fastened so that it cannot be unravelled again. In the mathematical sense, we prevent the knot from being unravelled by 'glueing' the ends of the string together to form a loop. We also think of the string as having no thickness (ie. a 1-dimensional object). What we are left with is a one-dimensional curve embedded in three-dimensional space that has no self intersections. If working from a geometrical point of view, the curve can be thought of as made up of a number of straight sections, joined end to end (a *polygonal* knot) but these can be made to be so small that we usually think of what is called a *smooth* knot.

Definition 1.1 *A* knot *K is a locally flat subset of points homeomorphic to a circle.*

In this definition, the condition of *local flatness* requires that at each point of the curve, within some arbitrarily small spherical neighbourhood of the point, the arc of the knot contained within the sphere is homeomorphic to a diameter of the sphere. (Intuitively: if we look at the curve of the knot close enough then each section looks flat). The reason for this constraint is to prevent the occurrence of entities called *wild knots*. These are knots that have knotted features at arbitrarily small scales in a similar way that detail can be found in fractal pictures however much the picture is zoomed in on. Incidentally, polygonal knots can never be wild so an alternative way around this problem is to only consider the class of polygonal knots.

Note that a knot is usually thought of as having an *orientation*. That is, we travel around the curve of the knot in a particular direction.

So, when are two knots the same? The definition above gives describes a knot as a set of points, but we want to think of two knots as equivalent even if they are not equivalent sets. Formally, we describe two knots as being equivalent (or of the same knot type) if they are *ambient isotopic*.

Definition 1.2 *An* isotopy *is a continuous map* $h : X \times [0,1] \to \mathbb{R}^3$ *where each* $h_t = h \mid X \times \{t\}$ *is one-to-one. By convention,* h_0 *is the identity map.*

Definition 1.3 *Two knots* K_1 *and* K_2 *are* ambient isotopic *if there is an isotopy* $h : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ *such that* $h(K_1, 0) = h_0(K_1) = K_1$ *and* $h(K_1, 1) = h_1(K_1) = K_2$.

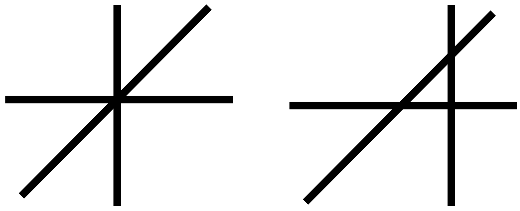
These definitions allow us to deform our knot in the expected manner: the arcs can be bent and moved through space without passing through one another, the entire knot can be shrunk or grown and we are not permitted to pull the knot so tight that it unknots itself by disappearing into a point.

1.2 Knot diagrams

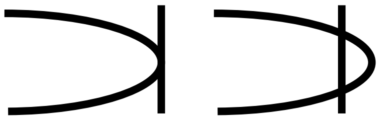
We now have an adequate definition of a knot in 3-space, but in order to work with knots more easily we want to be able to represent them in a diagram. Intuitively, we do this by projecting the knot downwards (casting a 'shadow') onto the plane and marking it in some way to show whether an arc is passing over or under another arc where there is a crossing. We also require, to prevent confusion, that all singularities are double points with the approaching arcs having distinct tangents (see Figure 1.2). A diagram satisfying these conditions is called a *regular projection* of the knot.

There are two common conventions for denoting which arc is the overpass and which the underpass. The easiest to understand is to show the underpass as a broken line and I shall use this convention in this document. In Alexander's paper, he marks the crossing point by placing two dots to the left hand side of the underpass as you follow the given orientation of the knot (marked by an arrow in the diagram). This convention is useful in later calculations and in those cases I will redundantly use both marking styles for clarity.

Notice that the orientation of the knot naturally leads to an orientation of a crossing point. Imagine that the knot is an electric train set and the trains move in the direction of the knot's orientation. The overpass is a bridge over another line. If we sit on the overpass, facing in the direction of the train, then the trains underneath will either pass from right to left or from left to right. In the first case we call the crossing *right handed* and in the second case we call it *left handed*.

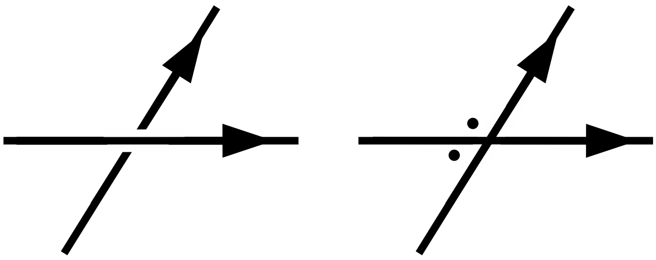


(a) Triple point singularity (b) Regular projection



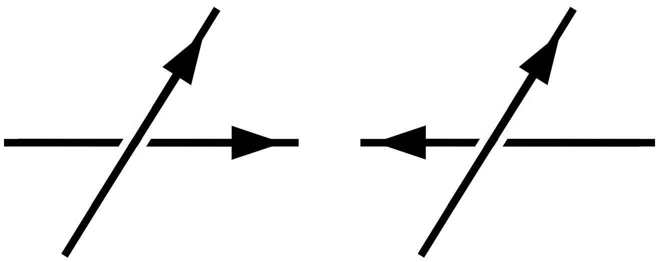
(c) Double point with indistinct tangents (d) Regular projection

Figure 1.2: Examples of illegal singularities



(a) Broken line (b) Alexander's notation

Figure 1.3: Styles of marking crossing points



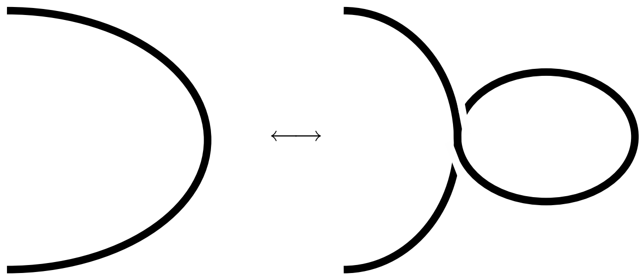
(a) Left-handed crossing (b) Right-handed crossing

Figure 1.4: Crossing points of a diagram

Although using diagrams makes it easier for us to visualise a knot, they introduce a complication in that projecting the knot from different angles will result in different diagrams for the knot. Of course we can allow the arcs of the diagrams to be continuously deformed but we have to define new rules because of our constraints on the types of singularity we allow.

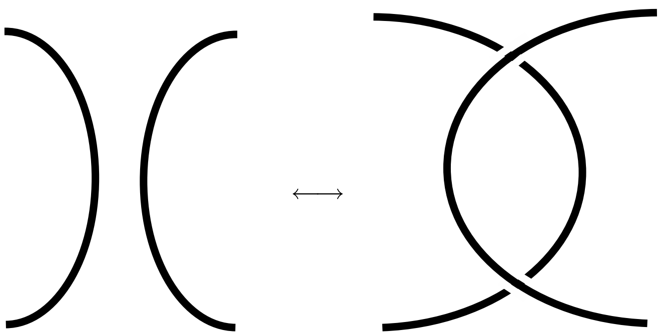
1.3 Reidemeister moves and knot invariants

Luckily, there are only a small number of cases in which deforming the knot results in illegal singularities. Specifically, there are three moves that can be made in the neighbourhood of a crossing point which do not alter the knot type. These are called *Reidemeister moves* after the German topologist and group theorist Kurt Reidemeister. It can be proved that whenever two diagrams represent equivalent knots, there exists a sequence of Reidemeister moves to transform one diagram into the other.



(a) Before move (b) After move

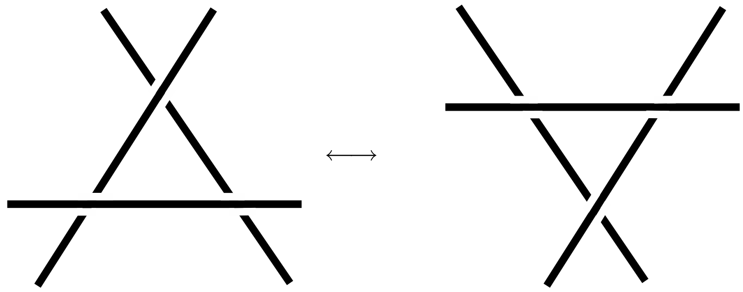
Figure 1.5: Reidemeister I move



(a) Before move (b) After move

Figure 1.6: Reidemeister II move

Reidemeister moves provide a tool for directly showing whether two knots are equivalent and can be easily applied in the case of simple knots with few crossings. With larger knots, however, simply trying moves to see if one knot diagram can be turned into another becomes very inefficient. What is required is an algorithmic method that can be used to prove, in a finite number of steps, that two knots are either of the same knot type or of different knot types. For this, we turn to *knot invariants.*



(a) Before move (b) After move

Figure 1.7: Reidemeister III move

Definition 1.4 *An* invariant *of an object, with respect to some transformation of the object, is some quantity or characteristic that does not change under the transformation.*

In the case of knots, an invariant is something that is not changed under ambient isotopy or, when dealing with diagrams, something that is not changed under any of the Reidemeister moves. This means that any two equivalent knots will have the same value for any particular invariant.

Since the number of Reidemeister moves is so small, it makes it relatively simple to prove whether something is an invariant or not: we need only to apply the three moves in turn and see whether it remains unchanged. This provides the motivation for most of the proofs in the rest of the document.

1.4 Links

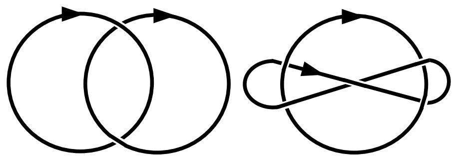
The title of Alexander's paper on which this project is based mentions another mathematical object: links. This generalises the idea of a knot to an entity with more than one component. I will give these only a brief treatment.

Definition 1.5 *A* link *is a finite disjoint union of knots:* $L = K_1 \cup \ldots \cup K_n$.

That is, a link has a number of *components*, each of which is a knot. We call the number of components of a link its *multiplicity* and so any knot is just a link of multiplicity 1.

The name *link* suggests that the components of the link are embedded in such a way that they cannot be pulled apart but, just as a knot need not necessarily be knotted, a link does not have to be linked. A link that is just n copies of the unknot sitting in 3-space is called the *trivial link* of multiplicity n.

Two examples of simple 2-component links are the Hopf link and the Whitehead link. Another interesting example is the Borromean rings. These are a 3-component link with the property that if any one of the rings (unknots) is removed then the remaining two rings become unlinked. Historically, this link appears as a heraldic symbol to represent the notion that the strength of a group of people depends on each of the individuals and the loss of any one would undermine the strength of the whole.



(a) Hopf link (b) Whitehead link

Figure 1.8: Two 2-component links

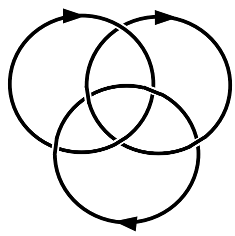


Figure 1.9: The Borromean rings

As in the case of knots, *link invariants* are employed in studying whether two links are equivalent. Again, two equivalent link diagrams can always be transformed into one another using the Reidemeister moves and so these can be used to verify our invariants.

We will briefly examine a link invariant, the *linking number*, of a pair of link components before moving on to the main focus of this document.

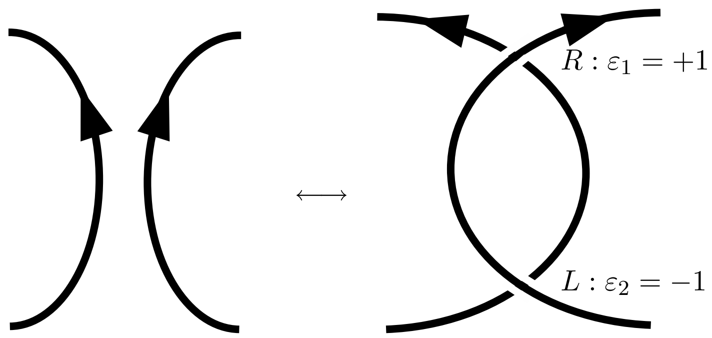
Definition 1.6 *Given an oriented link diagram D, choose two components of the link D_i, D_j. For each crossing c_r in which D_i and D_j cross, set $\varepsilon_r = +1$ if the link is right-handed and $\varepsilon_r = -1$ if the link is left-handed. Then, the* linking number *$lk(D_i, D_j)$ of the two components is the sum:*

$$\frac{1}{2}\sum_r \varepsilon_r$$

For example, our Hopf link has two right-handed crossings. So $\varepsilon_1 = \varepsilon_2 = +1$ and $lk(D_1, D_2) = \frac{1}{2}(1+1) = 1$. The Whitehead link has two right-handed crossings and two left-handed crossings so their sum is zero. Hence $lk(D_1, D_2) = 0$.

It is simple to demonstrate that the linking number is not changed by Reidemeister moves. First, a Reidemeister I move only creates a crossing point within one component so this crossing is not counted in the linking number.

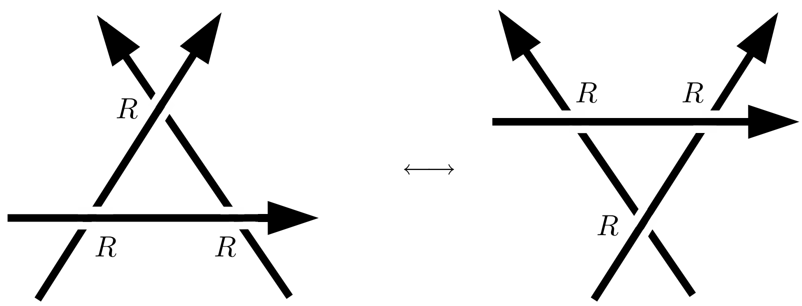
With a Reidemeister II move, assuming that the two arcs belong to two different link components, we either gain or lose one right-handed crossing and one left-handed crossing. These cancel each other out leaving the sum unaltered.



(a) Before move (b) After move

Figure 1.10: Reidemeister II move (on oriented link diagram)

With a Reidemeister III move, the diagram before the move has three right-handed crossings and also has three after the move. Hence, whichever components the arcs of the diagram belong to, the sum of the values ε_r is unchanged. This is not quite enough: the diagram only shows one possible orientation of the arcs. By changing the orientation of the arcs in the diagram, the handedness of the crossing points would be different but it is a simple matter to show that the sum of the ε_r's is the same in each case and so the linking number is unchanged.



(a) Before move (b) After move

Figure 1.11: Reidemeister III move (on oriented link diagram)

Since the linking number of two link components is unchanged under each of these moves, it is a link invariant. Hence, since the linking numbers of the Hopf link and the Whitehead link are different, we know that we cannot apply a sequence of Reidemeister moves to transform one diagram into the other and so the links cannot be equivalent.

In the remainder of this document, we will restrict our attention to knots, referring again to links only in the final chapter.

Chapter 2

The Alexander Polynomial: the combinatorial route

James Waddell Alexander was an American Mathematician, born in New Jersey in 1888. He studied Mathematics and Physics at Princeton University and was awarded his PhD in 1915. During World War I, Alexander contributed his mathematical proficiency by working with the military at a weapons testing site. During World War II he also worked at the US Air Forces Office of Scientific Research and Development. Alexander held various professorships at Princeton and was one of the first members of the Institute for Advanced Study. Being descended from the president of the Equitable Life Assurance Company, however, he had become a millionaire through inheritance and did not take salaries while in these positions. In the 1950s, the political environment under Joseph R McCarthy coupled with Alexander's left-wing political views brought him under suspicion and he became somewhat of a recluse, last appearing in public in 1954. He died in 1971.

Alexander's main contribution to knot theory was a polynomial invariant that can be calculated from the diagram of a knot. In overview: each crossing point of the diagram yields an equation in variables r_i. These equations can then be represented in a matrix from which we can derive a polynomial by operating on the matrix and taking the determinant. The resulting polynomial in powers of t must then be normalised, and it is this normalised polynomial which is invariant for equivalent knots.

This chapter gives an outline of the steps involved in calculating the polynomial in the manner given in Alexander's paper and demonstrates the calculation in the cases of the knots 3_1 and 5_2 . All required results are proved in Chapter 3.

2.1 Calculating the Alexander Polynomial

We start our process with an oriented diagram D of a knot K. Let there be v crossing points of the diagram: $c_1, c_2, \ldots, c_v$. Then, by Eulers theorem, it follows that the arcs of the diagram divide the plane up into $v + 2$ regions (including the region outside of the knot). We label the regions $r_0, r_1, \ldots, r_{v+1}$.

We denote the underpasses of the diagram with the second convention mentioned in Chapter 1: the two dots to the left hand side of the underpass. Now consider an arbitrary crossing point, c_i.

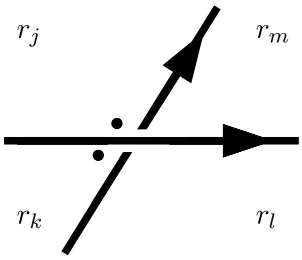


Figure 2.1: A dotted crossing point

Let the four regions surrounding it be r_j, r_k, r_l and r_m where we go around the crossing point anticlockwise and where the dots lie in regions r_j and r_k. We can now define the linear equation:

$$c_i(r) = tr_j - tr_k + r_l - r_m = 0$$

by taking an alternating sum of the symbols representing the four regions in their cyclic order and multiplying the dotted regions by t.

Defining such an equation for each of the crossings in the diagram yields a system of v equations in $v+2$ variables, which we can then represent in a $v \times (v+2)$ matrix, M, where each entry is either $\pm t$, ± 1 or 0. In the matrix constructed as just described, each row of the matrix corresponds to a crossing point of the diagram and each column corresponds to a region. The next step in this process is to choose two neighbouring regions r_p, r_q and delete their respective columns v_p, v_q from the matrix. Any two neighbouring regions may be chosen and it is proved in the next chapter that the regions chosen will not affect the resulting invariant.

Deleting columns v_p, v_q leaves us with a square $v \times v$ matrix, $M_{p,q}$. The matrix $M_{p,q}$ is called the *Alexander matrix* of the knot K. Now let $\Delta_{p,q}(t)$ be the determinant of this square matrix, which will be a polynomial in powers of t with integer coefficients.

Theorem 2.1 *The polynomial $\Delta_{p,q}(t)$ obtained as described above, computed from any other equivalent knot diagram of K differs only by a factor of $\pm t^k$ for some integer k.*

This theorem is proved in the following chapter.

The fact that the obtained polynomial may differ by a factor of $\pm t^k$ when computed from a different diagram of the knot suggests that we need some normal form so that a unique polynomial can be associated to each knot. One possible form is setting $\Delta_K(t) = \pm t^n \Delta_{p,q}(t)$ so that the term of lowest degree in $\Delta_K(t)$ is a positive constant. This is the required normal form which gives us our knot invariant and is called the *Alexander polynomial*.

2.2 The Alexander polynomial of 3_1

Consider the diagram of the trefoil. Examining crossing c_1 we see that regions r_2 and r_0 are dotted and that the anticlockwise cyclic order is r_0, r_3, r_4, r_1.

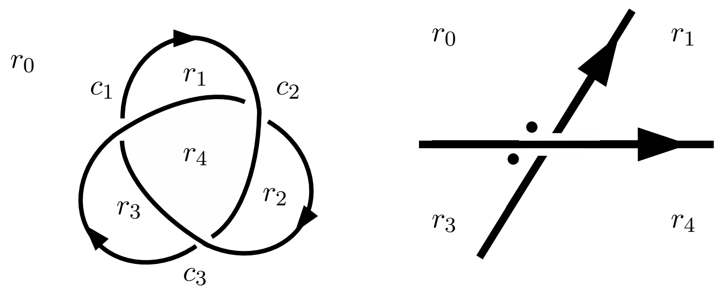


Figure 2.2: Crossing c_1 of the trefoil knot

This yields the equation:

$$c_1(r) = tr_0 - tr_3 + r_4 - r_1 = 0$$

Repeating the same process for crossing points c_2 and c_3 gives us the remaining equations:

$$c_2(r) = tr_0 - tr_1 + r_4 - r_2 = 0$$

$$c_3(r) = tr_0 - tr_2 + r_4 - r_3 = 0$$

Altogether, we represent these equations in the matrix:

$$M = \begin{pmatrix} t & -1 & 0 & -t & 1 \\ t & -t & -1 & 0 & 1 \\ t & 0 & -t & -1 & 1 \end{pmatrix}$$

Two neigbouring regions are r_3 and r_4 so we delete the last two columns of the matrix and take the determinant of the square matrix $M_{3,4}$:

$$\begin{aligned} \Delta_{3,4}(t) = \det(M_{3,4}) = \begin{vmatrix} t & -1 & 0 \\ t & -t & -1 \\ t & 0 & -t \end{vmatrix} &= t \begin{vmatrix} -t & -1 \\ 0 & -t \end{vmatrix} + \begin{vmatrix} t & -1 \\ t & -t \end{vmatrix} \\ &= t^3 - t^2 + t \\ &= t(1 - t + t^2) \end{aligned}$$

We then take out the factor of t to give the normalised polynomial:

$$\Delta_K(t) = 1 - t + t^2$$

This is the standard Alexander polynomial for the trefoil knot and so, by Theorem 2.1, calculating Δ_K from any other diagram of the trefoil will give the same answer.

2.3 The Alexander polynomial of 5_2

The process for this knot follows the exact steps as for the trefoil but is made more complicated by the larger number of crossing points, which lead to a bigger Alexander matrix. Again, examine the crossing c_1 in the diagram. The regions r_1 and r_2 are dotted and the cyclic order of the regions surrounding the crossing is r_1, r_2, r_3, r_0.

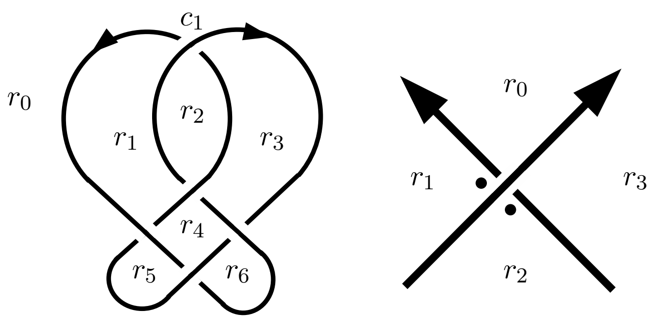


Figure 2.3: Crossing c_1 of 5_2

Hence the equation derived is:

$$c_1(r) = tr_1 - tr_2 + r_3 - r_0 = 0$$

Applying the same process to the crossings $c_2, \ldots, c_5$ yields the matrix:

$$M = \begin{pmatrix} -1 & t & -t & 1 & 0 & 0 & 0 \\ 0 & t & -1 & 1 & -t & 0 & 0 \\ -t & t & 0 & 0 & -1 & 1 & 0 \\ -t & 0 & 0 & 1 & -1 & 0 & t \\ -1 & 0 & 0 & 0 & -t & 1 & t \end{pmatrix}$$

In this diagram, we choose neighbouring regions r_4 and r_5 and delete their columns to give the square matrix $M_{4,5}$. Then:

$$\Delta_{4,5}(t) = \det(M_{4,5}) = \begin{vmatrix} -1 & t & -t & 1 & 0 \\ 0 & t & -1 & 1 & 0 \\ -t & t & 0 & 0 & 0 \\ -t & 0 & 0 & 1 & t \\ -1 & 0 & 0 & 0 & t \end{vmatrix} = \begin{vmatrix} -1 & 0 & 1-t & 0 & 0 \\ 0 & t & -1 & 1 & 0 \\ -t & t & 0 & 0 & 0 \\ -t & 0 & 0 & 1 & t \\ -1 & 0 & 0 & 0 & t \end{vmatrix}$$

$$= -\begin{vmatrix} t & -1 & 1 & 0 \\ t & 0 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & t \end{vmatrix} + (1-t)\begin{vmatrix} 0 & t & 1 & 0 \\ -t & t & 0 & 0 \\ -t & 0 & 1 & t \\ -1 & 0 & 0 & t \end{vmatrix}$$

$$= -\left\{ t\begin{vmatrix} 0 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & t \end{vmatrix} + \begin{vmatrix} t & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & t \end{vmatrix} + \begin{vmatrix} t & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & t \end{vmatrix} \right\}$$

$$+(1-t)\left\{ (-t)\begin{vmatrix} -t & 0 & 0 \\ -t & 1 & t \\ -1 & 0 & t \end{vmatrix} + \begin{vmatrix} -t & t & 0 \\ -t & 0 & t \\ -1 & 0 & t \end{vmatrix} \right\}$$

$$= -t^2 + (1-t)(-t^2 + 2t^3)$$

$$= -2t^3 + 3t^3 - 2t^4$$

We then normalise the polynomial by dividing by a factor of $-t^2$ to give the polynomial:

$$\Delta_K(t) = 2 - 3t^2 + 2t^2$$

You can see that the polynomial for 5_2 is different from the polynomial for 3_1. After the next chapter, we will be able to use this fact to prove that the two knots are of different types.

Chapter 3

The invariance of the Alexander polynomial

At the end of the previous chapter, we demonstrated that a polynomial can be calculated from a diagram of a knot and that the polynomials calculated from the knots 3_1 and 5_2 are different. In this chapter, we show that a polynomial calculated in such a way is an invariant of a knot and hence knots with different Alexander polynomials are necessarily of distinct knot types.

The proof follows that given in Alexander's 1927 paper *Topological invariants of knots and links*, and the argument centres on defining an equivalence between matrices and showing first that equivalent diagrams lead to equivalent matrices and then that equivalent matrices have determinants which differ only by powers of $\pm t^k$. Hence, when normalised, the Alexander polynomial is invariant.

3.1 The index of a region

Alexander assigns an integer to each region of the knot diagram called the *index* of the region. These integers are determined by assigning any integer p to a chosen region and then determining the indices of the remaining regions by setting an index to $p + 1$ if we cross into the region from right to left (with respect to the orientation of the diagram) and to $p - 1$ if we cross from left to right.

Clearly, since all regions can be reached by crossing over the arcs of the diagram, this process determines the indices of all the regions of the diagram. Also the process will always produce a consistent indexing.

Consider now the crossing points of the diagram. Clearly, at each point there will be two regions with the same index, say p, one of index $p+1$ and one of index $p-1$.

At a left-handed crossing, the first dotted region is of index p and the second is of index $p + 1$. At a right-handed crossing, the first dotted region is of index $p + 1$ and the second is of index p (recall the cyclic order is anticlockwise).

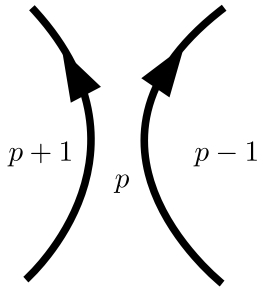
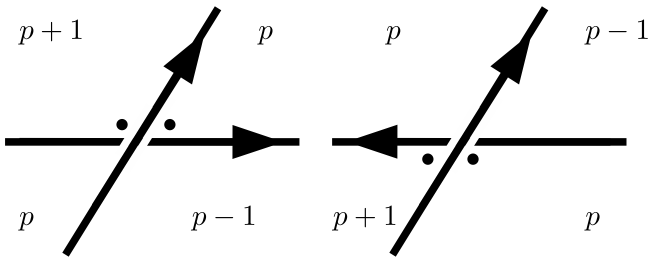


Figure 3.1: Indexed regions



(a) Left-handed crossing (b) Right-handed crossing

Figure 3.2: Indices around crossing points

Since each region has an index associated to it, when the equations of the diagram are represented in a matrix each column of the matrix also has a corresponding index.

3.2 Obtaining the square matrix

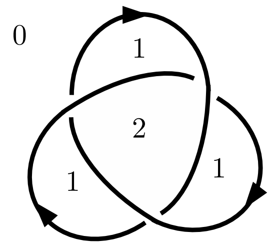
Recall that our process of finding the Alexander polynomial involved deleting two columns from the matrix corresponding to adjacent regions of the diagram. By the indexing process, any two adjacent regions will have indices differing by 1 and, in fact, any two columns with indices differing by 1 may be deleted.

Proposition 3.1 *If we reduce M to a square matrix $M_{p,q}$ by deleting two of its columns of index p and $p+1$ then the determinants of the two matrices will differ only by a factor of $\pm t^k$ for any two such columns.*

To prove this claim, let R_p denote the sum of all columns of index p. Then, since each row of the matrix has one t, one $-t$, one 1 and one -1, we have:

$$\sum_p R_p = 0$$

where 0 denotes the column of zeroes. For example, in the case of the trefoil:



We set the index of r_0 to be zero and then apply the indexing rules to find that r_1, r_2 and r_3 all have index 1 and r_4 has index 2. Then:

$$R_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad R_1 = \begin{pmatrix} t+1 \\ t+1 \\ t+1 \end{pmatrix}, \quad R_0 = \begin{pmatrix} -t \\ -t \\ -t \end{pmatrix}$$

And so $R_2 + R_1 + R_0 = 0$.

Now multiply each column of index p by a factor t^{-p}. Since each row of the matrix corresponds to a crossing point; and at each crossing point the indices of the regions is determined, we have:

$$c_L(r) = t.t^{-p} r_j - t.t^{-(p+1)} r_k + t^{-p} r_l - t^{-(p-1)} r_m$$

in the case of a left-handed crossing and:

$$c_R(r) = t.t^{-(p+1)} r_j - t.t^{-p} r_k + t^{-(p-1)} r_l - t^{-p} r_m$$

in the case of a right-handed crossing. Clearly, in both cases the sum of the coefficients is zero and so the sum of the columns of the matrix will again be the zero vector. ie.

$$\sum_p t^{-p} R_p = 0$$

And so we can combine the above two sums to give:

$$\sum_p (t^{-p} - 1) R_p = 0$$

Note that since $t^0 = 1$, the terms in R_0 in the sum cancel each other out. Hence we see, from the above sum, that if r_j is a region of index p with corresponding column v_j then $(t^{-p} - 1)v_j$ is expressible as a linear combination of the other columns with nonzero index. Also, the coefficients of the columns in the linear combination are of the form $-(t^{-q} - 1)$ for each column of index q.

Now consider the matrices $M_{0,j}$ and $M_{0,k}$ where the columns v_j and v_k have indices p and q respectively. Because of the above result and by properties of determinants, we see that:

$$(t^{-q} - 1)\Delta_{0,j}(t) = \pm(t^{-p} - 1)\Delta_{0,k}(t)$$

Then, since the indices of the regions are determined only up to an additive constant (we can set the initial p to be any number we choose), if v_l and v_m are two more columns of M of index r and s respectively then we obtain the relations:

$$(t^{r-q}-1)\Delta_{l,j}(t) = \pm(t^{r-p}-1)\Delta_{l,k}(t),$$
$$(t^{q-s}-1)\Delta_{k,l}(t) = \pm(t^{q-r}-1)\Delta_{k,m}(t)$$

which we can combine to give:

$$\Delta_{l,j}(t) = \pm\frac{(t^{q-r})(t^{r-p}-1)}{t^{q-s}-1}\Delta_{k,m}(t)$$

Finally, setting $p = r+1$ and $s = q+1$ we obtain:

$$\Delta_{l,j} = \pm t^{q-r}\Delta_{k,m}$$

ie. Whenever we remove two columns from the matrix of consecutive index, the determinant of the resulting matrix differs by $\pm t^{q-r}$, proving the proposition.

3.3 ϵ-equivalent matrices

Different diagrams of the same knot will give different matrices when we apply the procedures outlined in Chapter 2. So we need a way of defining an equivalence between matrices so that a knot always yields a matrix in the same equivalence class.

Definition 3.2 *Two matrices*[1] M_1 *and* M_2 *are* ϵ-equivalent *if it is possible to transform one into the other by a sequence of the following operations:*
(α) Multiplying a row or column by -1
(β) Swapping two rows or columns
(γ) Adding one row or column to another
(δ) Either adding or removing a border where the corner element is 1 and all other elements are 0, as shown below:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \overset{\sim}{\longleftrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix}$$

(ϵ)Multiplying or dividing a column by t

By properties of determinants, it is simple to verify that the operations α–ϵ will change the determinant of a matrix by at most a factor of $\pm t^k$. Hence any two ϵ-equivalent matrices have determinants which differ by at most a factor of $\pm t^k$.

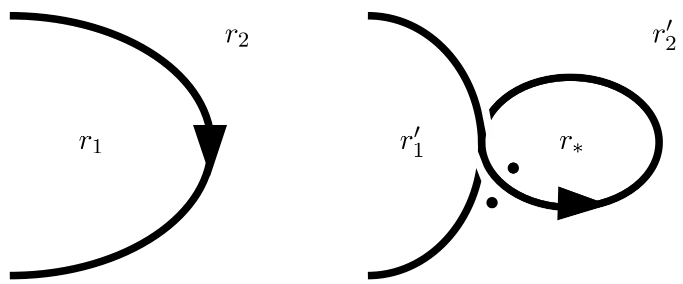
Recall that if two diagrams of knots are equivalent, then one diagram can be transformed into the other via a sequence of Reidemeister moves. We use these to show that equivalent knots have ϵ-equivalent Alexander matrices.

[1] With entries which are polynomials in t with integer coefficients

Theorem 3.3 *If two diagrams D_1, D_2 represent knots of the same type then their square matrices M_1, M_2 are ϵ-equivalent.*

We prove this by looking at the effect of the Reidemeister moves on the matrix of the diagram.

(I) The diagram begins with regions $r_1, r_2, \ldots$ and the formation of a loop creates a new region r_* and adds a new crossing point to the diagram.



(a) Before move (b) After move

Figure 3.3: Reidemeister I move

Say the Alexander matrix of the knot before the transformation is M. The Reidemeister I move has the effect then of adding a new row and a new column to M. This new matrix will have the form:

$$\begin{array}{c}
\begin{array}{cccc} r_* & r_1' & r_2' & \cdots \end{array} \\
\begin{pmatrix} -t & -1 & t+1 & 0\cdots 0 \\ 0 & & & \\ \vdots & & M & \\ 0 & & & \end{pmatrix}
\end{array}$$

Since regions r_1' and r_2' are adjacent we may delete these from the matrix without affecting the knot invariant. This leaves us with the matrix:

$$\begin{array}{c}
\begin{array}{cc} r_* & \cdots \end{array} \\
\begin{pmatrix} -t & 0\cdots 0 \\ 0 & \\ \vdots & M_{1,2} \\ 0 & \end{pmatrix}
\end{array}$$

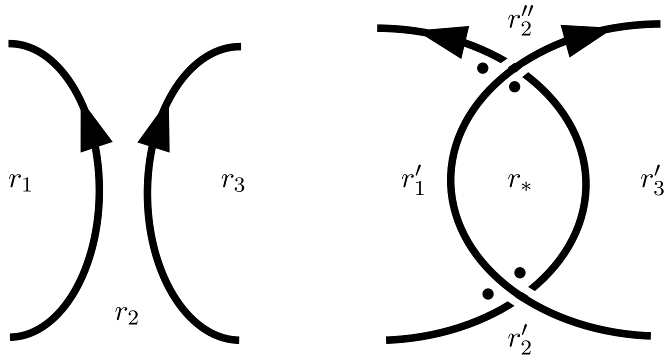
We can then divide the r_* column by t (operation ϵ) and multiply by -1 (operation α). This leaves us with the matrix:

$$\begin{array}{c}
\begin{array}{cc} r_* & \cdots \end{array} \\
\begin{pmatrix} 1 & 0\cdots 0 \\ 0 & \\ \vdots & M_{1,2} \\ 0 & \end{pmatrix}
\end{array}$$

Finally, removing the border using operation δ leaves us with $M_{1,2}$: a viable square matrix for the Alexander polynomial of the original knot. Hence Δ_K is invariant under Reidemeister I moves.

Note that the diagram I use for the Reidemeister moves could be oriented or dotted differently, resulting in slightly different matrices. The methods outlined, however, will be similar in all cases and a full treatment is omitted for the sake of space.

(II) In this case, we begin with a diagram with regions $r_1, r_2, r_3, \ldots$ and the transformation creates another two crossing points, a new region r_* and splits the region r_2 into r_2' and r_2''.



(a) Before move (b) After move

Figure 3.4: Reidemeister II move

The matrix after the transformation will have the form:

$$\begin{array}{c}
\begin{array}{cccccc} r_* & r_1' & r_2' & r_2'' & r_3 & \cdots \end{array} \\
\begin{pmatrix}
-t & t & 0 & -1 & 1 & 0\cdots0 \\
t & -t & 1 & 0 & -1 & 0\cdots0 \\
0 & | & | & | & & \\
\vdots & u & v & w & & M_{1,2} \\
0 & | & | & | & &
\end{pmatrix}
\end{array}$$

In this matrix, u is the column for r_1 in the original matrix and the entries for r_2 are divided between v and w (since the region has been divided in two). We shall choose to delete the columns corresponding to regions r_1' and r_2''. This leaves us with:

$$\begin{pmatrix}
-t & 0 & 1 & 0\cdots0 \\
t & 1 & -1 & 0\cdots0 \\
0 & | & & \\
\vdots & v & & M_{1,2} \\
0 & | & &
\end{pmatrix}$$

We may divide the first column by t and then add the first row to the second to cancel the entries to get:

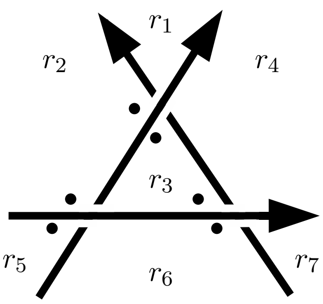
$$\begin{pmatrix} -1 & 0 & 1 & 0\cdots 0 \\ 0 & 1 & 0 & 0\cdots 0 \\ 0 & | & & \\ \vdots & v & & M_{1,2} \\ 0 & | & & \end{pmatrix}$$

Then add column r_* to r_3', multiply column r_* by -1 and remove the border to get:

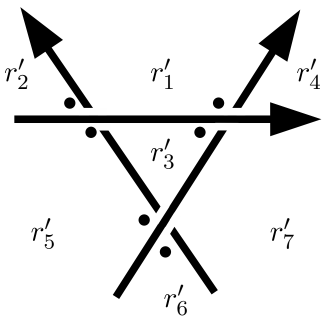
$$\begin{pmatrix} 1 & 0 & 0\cdots 0 \\ | & & \\ v & & M_{1,2} \\ | & & \end{pmatrix}$$

We may cancel all of the entries of v using multiple applications of operations α–ϵ. This is done by multiplying row 1 of the matrix by the appropriate power of t, adding row 1 to another row so that the entry in column 1 cancels and then dividing row 1 by the same power of t so it keeps a 1 in the first entry. This will then leave us with $M_{1,2}$ bordered as described in operation δ and we can remove the border to leave us with $M_{1,2}$ again.

(III)In the case of a Reidemeister III move, the number of regions is unchanged but the entries around the crossing points differ.



(a) Before move



(b) After move

Figure 3.5: Reidemeister III move

Before the transformation, the matrix obtained from the diagram will have the form:

$$M = \begin{array}{c} \begin{array}{cccccccc} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & \cdots \end{array} \\ \begin{pmatrix} -1 & t & -t & 1 & 0 & 0 & 0 & 0\cdots 0 \\ 0 & t & -1 & 0 & -t & 1 & 0 & 0\cdots 0 \\ 0 & 0 & t & -1 & 0 & -t & 1 & 0\cdots 0 \\ | & | & 0 & | & | & | & | & \\ u & v & \vdots & w & x & y & z & X \\ | & | & 0 & | & | & | & | & \end{pmatrix} \end{array}$$

where X is the remaining portion of the matrix defined by the procedure in Chapter 2. After the transformation, the resulting matrix is:

$$M' = \begin{array}{c} \begin{matrix} r'_1 & r'_2 & r'_3 & r'_4 & r'_5 & r'_6 & r'_7 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 0 & -1 & 0 & t & -t & 1 & 0\cdots 0 \\ t & 0 & -t & -1 & 0 & 0 & 1 & 0\cdots 0 \\ -1 & t & 1 & 0 & -t & 0 & 0 & 0\cdots 0 \\ | & | & 0 & | & | & | & | & \\ u & v & \vdots & w & x & y & z & X \\ | & | & 0 & | & | & | & | & \end{pmatrix} \end{array}$$

Since this example involves such large matrices, we will use a convenient result to simplify them.

Proposition 3.4 *The matrix N obtained by changing the signs of all the negative elements of M is ϵ-equivalent to the matrix M.*

To see why this is true, recall the arrangement of the indices of the regions around each crossing. As you go around the crossing, the indices will alternate between odd and even. Hence, as entries in the matrix, the odd regions at a crossing will either both be positive or both negative (and the corresponding even regions will have the opposite parity). So if we multiply each odd column by -1 then each row will have only positive entries or only negative entries. Finally we can multiply all negative rows by -1 to give an entirely positive matrix. Note also that this process is reversible so we can recover our original matrix M from the positive matrix N.

Applying this result to M gives us:

$$N = \begin{array}{c} \begin{matrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & \cdots \end{matrix} \\ \begin{pmatrix} 1 & t & t & 1 & 0 & 0 & 0 & 0\cdots 0 \\ 0 & t & 1 & 0 & t & 1 & 0 & 0\cdots 0 \\ 0 & 0 & t & 1 & 0 & t & 1 & 0\cdots 0 \\ | & | & 0 & | & | & | & | & \\ u' & v' & \vdots & w' & x' & y' & z' & X' \\ | & | & 0 & | & | & | & | & \end{pmatrix} \end{array}$$

Here w'–z' are the columns w–z with the signs of all negative elements changed (analogously for X') and we aim to find a sequence of operations α–ϵ that will transform this matrix into:

$$N' = \begin{array}{c} \begin{matrix} r'_1 & r'_2 & r'_3 & r'_4 & r'_5 & r'_6 & r'_7 & \cdots \end{matrix} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & t & t & 1 & 0\cdots 0 \\ t & 0 & t & 1 & 0 & 0 & 1 & 0\cdots 0 \\ 1 & t & 1 & 0 & t & 0 & 0 & 0\cdots 0 \\ | & | & 0 & | & | & | & | & \\ u' & v' & \vdots & w' & x' & y' & z' & X' \\ | & | & 0 & | & | & | & | & \end{pmatrix} \end{array}$$

To save space, in the following calculation I will display only the first three rows and first seven columns of N but we must take into account the nature of the rest of the matrix. To avoid changing the entries of $w'–z'$, only the column r_3 is permitted to be added to the other columns or multiplied by -1 or t. We begin by swapping rows 1 and 3:

$$\begin{pmatrix} 1 & t & t & 1 & 0 & 0 & 0 \\ 0 & t & 1 & 0 & t & 1 & 0 \\ 0 & 0 & t & 1 & 0 & t & 1 \end{pmatrix} \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 1 & 0 & t & 1 \\ 0 & t & 1 & 0 & t & 1 & 0 \\ 1 & t & t & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Times row 2 by } -t \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 1 & 0 & t & 1 \\ 0 & -t^2 & -t & 0 & -t^2 & -t & 0 \\ 1 & t & t & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Add row 1 to row 2} \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 1 & 0 & t & 1 \\ 0 & -t^2 & 0 & 1 & -t^2 & 0 & 1 \\ 1 & t & t & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Divide column 3 by } t \quad \rightarrow \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & t & 1 \\ 0 & -t^2 & 0 & 1 & -t^2 & 0 & 1 \\ 1 & t & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Subtract column 3 from column 4} \quad \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & t & 1 \\ 0 & -t^2 & 0 & 1 & -t^2 & 0 & 1 \\ 1 & t & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Times column 3 by } t \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 0 & 0 & t & 1 \\ 0 & -t^2 & 0 & 1 & -t^2 & 0 & 1 \\ 1 & t & t & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Add } t \text{ times row 3 to row 2} \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 0 & 0 & t & 1 \\ t & 0 & t^2 & 1 & -t^2 & 0 & 1 \\ 1 & t & t & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Add column 3 to column 5} \quad \rightarrow \begin{pmatrix} 0 & 0 & t & 0 & t & t & 1 \\ t & 0 & t^2 & 1 & 0 & 0 & 1 \\ 1 & t & t & 0 & t & 0 & 0 \end{pmatrix}$$

$$\text{Divide column 3 by } t \quad \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & t & t & 1 \\ t & 0 & t & 1 & 0 & 0 & 1 \\ 1 & t & 1 & 0 & t & 0 & 0 \end{pmatrix}$$

This is our required matrix N'.

Then, by the result quoted above, we can recover the original matrix M' by multiplying the appropriate rows and columns by factors of -1. We can then choose any two neighbouring regions and delete their columns from the matrices to find the respective determinants. This shows that the determinant of the Alexander matrix is not changed by a Reidemeister III move of the type illustrated in the figure. Again, other orientations of the arcs are possible but the method of proof would be the same as above in all cases[2].

In overview then, we have demonstrated that, given a particular diagram of a knot, we will always derive the same polynomial $\Delta_K(t)$, whichever columns we choose to omit from the matrix. We have also shown that any other diagram from the same knot will lead to the same polynomial since two diagrams of the same knot can always be transformed into one another by a series of Reidemeister moves. By the above proof, whenever we perform a Reidemeister move on a diagram the resulting square matrix will be ϵ-equivalent to the original.

This proof validates the demonstration in the previous chapter that the knots 3_1 and 5_2 are topologically distinct. In fact, the Alexander polynomial is different for all prime knots with eight or fewer crossings. If you allow knots with a larger number of crossings, however, we begin to find repetitions of the same Alexander polynomial and so it cannot be used to distinguish between knots with a higher number of crossings.

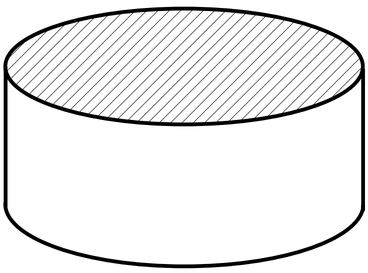
It should also be noted that the Alexander polynomial does not distinguish *handedness*. The trefoil used in the above example is a *right-handed trefoil*, so called because all of its crossing points are right handed. There exists a corresponding *left-handed trefoil* (a mirror image of the right-handed trefoil) but the Alexander polynomial for this knot can be shown to be the same as for the right handed case.

[2] The proof above is my own. Alexander gives a shorter proof in his paper but it requires a certain amount of work by the reader to verify. I wanted to give an explicit sequence of operations α–ϵ in my demonstration which may take up a lot of space but is easier for the reader to check.

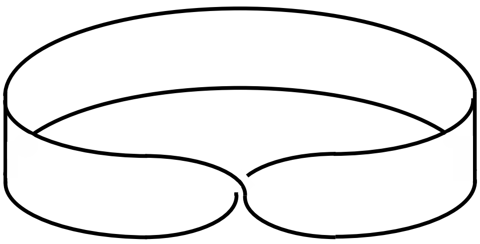
Chapter 4

Seifert surfaces: the geometric route

So far we have constructed our knots explicitly, but knots also arise naturally in other situations such as the closure of braids or random walks or, importantly for this chapter, the boundaries of *surfaces*. Here, we refer to a surface in the mathematical sense as a two dimensional manifold. A surface may be *orientable* or *nonorientable* and may be with or without boundary. Although a surface has no thickness, it is intuitively helpful to think of orientable surfaces as those that have two sides (which we could paint two different colours). A sphere is an example of an oriented surface without boundary (we can paint the inside red and the outside blue). A cylinder is another oriented example, but with a boundary consisting of two circles. A Möbius band (a strip connected end-to-end with one half-twist) has a boundary and is nonorientable; if we begin to paint in red then we are forced to cover the entire strip in red. The boundary has one component which looks like a circle with a twist.



(a) A cylinder

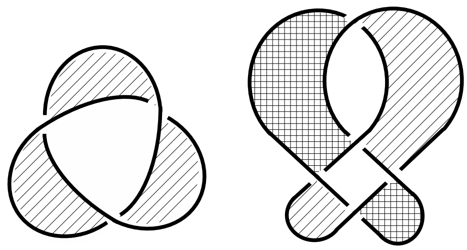


(b) A Möbius band

Figure 4.1: Examples of orientable and nonorientable surfaces

Knots occur as the boundaries of both orientable and nonorientable surfaces. It is easy to construct a surface from the diagram of the knot by producing what are known as *checkerboard colourings*. Begin by colouring a region adjacent to the 'outside' of the diagram in red and then follow over the crossing points to the opposite region at the crossing. Alternately colour the regions you reach in red or leave them uncoloured. If the entire knot is coloured at the end of this process then the surface is nonorientable. If there are uncoloured regions left then these can be coloured in blue; in this case the surface is orientable. (NB. regions that can be reached by crossing over from the outside of the knot should remain uncoloured and are interpreted as empty space).

For example, in the case of the trefoil, regions r_1, r_2 and r_3 are all coloured red and r_4 is left uncoloured. The surface we form looks like a band with three half-twists and, since only one colour is used, it is nonorientable. In the case of 5_2, we start by colouring r_3 and r_5 red but r_1 and r_6 are still uncoloured. We then colour these blue. The regions r_2 and r_4 appear as empty space and are left uncoloured. Since two colours are used, this surface is orientable. It looks like two discs connected by two bands with one half-twist and one band with three half-twists.



(a) The trefoil (b) The knot 5_2

Figure 4.2: Checkerboard colourings of knots

In 1930, Frankl and Pontrjagin proved that for all knots there exists a connected orientable surface with the knot as its boundary. The German mathematician Herbert Seifert improved on this result in 1934 by giving a separate proof which also included an algorithm for creating such a surface. An orientable surface with a given knot as its boundary is now called a *Seifert surface* and it should be noted that for one knot there is associated more than one Seifert surface, since the exact nature of the surface depends on which diagram of the knot is used.

4.1 Seifert's algorithm

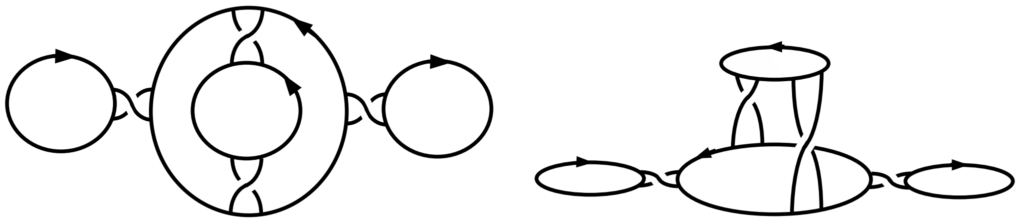
The following algorithm applied to a knot K gives a Seifert surface F for the knot:

Algorithm 4.1
(1) Choose an oriented diagram D for the knot.
(2) Beginning somewhere on the curve of the diagram, trace the orientation until a crossing point is reached.
(3) Switch to the other arc at the crossing point, still following the orientation.
(4) Repeat (3) until a closed loop is formed.
(5) Repeat (2)-(4) until all arcs of the diagram are traced, leaving a collection of the circles in the plane.

(6) Fill the circles in to form discs and connect them by bands with half-twists that correspond to the direction of the original crossing point in the knot diagram.

Remarks 4.2

After step (5) we are left with a collection of oriented circles. Some of these will lie next to each other and some will be nested within each other. When creating the surface, we imagine the nested circle as lying on a higher level than the outer circle, with the bands connecting across the two levels. Note that if two circles in the plane are connected by a band then their boundaries will have opposite orientations (ie. one clockwise and one anticlockwise) but if a circle is connected to a circle one level up then their boundaries will have the same orientation. It is impossible for circles to be directly connected across more than one level.



(a) Circles in the plane (b) Circles on two levels

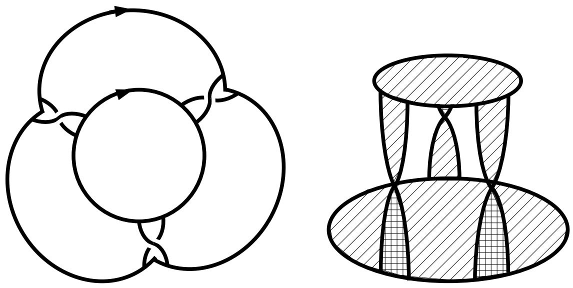
Figure 4.3: Raising nested circles to a higher level

Theorem 4.3 *Every knot is the boundary of an orientable surface.*

We give an intuitive proof using the idea of an orientable surface having two sides that we can paint red and blue. The algorithm produces a surface with the original knot as its boundary. Then we can use the property noted in the remarks to colour the discs with two colours. View the stack of planes from above: for all of the discs in each level, if the orientation of the circle on its boundary is clockwise then colour the 'top' of the disc red. If the orientation is anticlockwise then colour it blue. Finally, colour the 'bottoms' of the discs with the opposite colour to the tops. We see that we have consistently coloured the whole surface with two colours, since if we start on the red side of a disc and move on the same level to another disc then the half-twist takes us to the bottom of the next disc, which is also red. If we move up a level, then the half-twist keeps us on the top of the next disc, which is red. Hence the red sides of the discs form one whole side of the Seifert surface. Similarly the blue sides form the other side and the surface is orientable as required.

4.2 Seifert surfaces for 3_1 and 5_2

If we follow Seifert's algorithm for 3_1 then we form first a circle that goes around the edge of the whole knot and then a second circle inside the first. Hence we fill the circles in, raise the smaller disc to a higher level and join the two together with three bands (one for each crossing point). Each disc has a clockwise oriented boundary and each band has one right-handed half-twist.



(a) 3_1 after the Seifert algorithm

(b) As a surface shaded with two colours

Figure 4.4: A Seifert surface for the trefoil

In the case of 5_2, tracing the orientation of the knot as described in the algorithm results in four circles that lie next to each other in the plane. We fill these in to form discs. The top two are then connected by two bands, each with a right-handed half-twist and the bottom two are part of a chain that link the top two discs together. (We will later see that structures like this chain can be simplified).

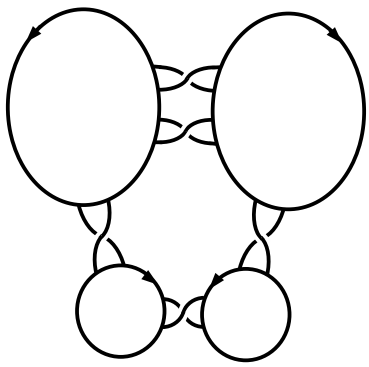


Figure 4.5: A Seifert surface for 5_2

4.3 Seifert matrices

From the Seifert surface F formed using the algorithm in the previous section, it is possible to construct a matrix from which we can derive the Alexander polynomial. The construction of the matrix involves loops in the Seifert surface which form a basis for a structure called the *first homology group*, $H_1(F)$, of the surface. The theory involved is beyond the scope of this document but I will give a description of the methods used to form the matrix, and demonstrate that using these methods on the knots 3_1 and 5_2 result in the same Alexander polynomial as calculated in Chapter 2.

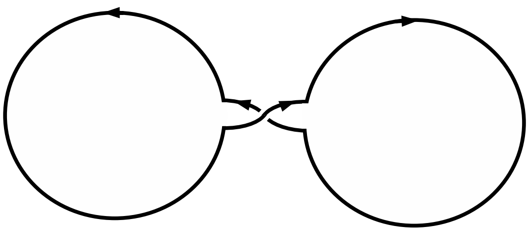
4.4 Simplifying the Seifert surface

We are looking for n loops that lie in F that form a basis for $H_1(F)$. From the information we get from these loops, we can form a $n \times n$ matrix, with each row and column corresponding to one of the loops.

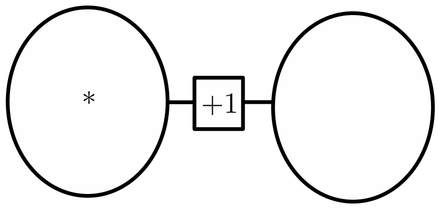
It can be shown that a Seifert surface made up of a number of discs and bands can be transformed into a single disc with a number, say n, of bands that connect back to itself (the final picture rather resembles a $2n$-legged octopus with its legs glued together in pairs). It can also be shown that if we fix some point in that disc and define n loops by paths which go down the centre of the bands and come back to the starting point, then those loops form a basis for $H_1(F)$ and hence are sufficient for our purposes.

To produce this simplified diagram of the surface we first put all of the discs onto the bottom level. In doing this we may have to allow some of the connecting bands to cross each other and it is important to keep track of which passes over which.

To make manipulation of the diagrams easier I have devised my own system of notation, which I will use in the examples. Each disc is represented by a circle and a band between discs by a line. In the middle of the line, to show the direction of the twist I place a box containing a $+1$ if the original crossing of the knot was right handed and a -1 if it was left handed. I mark the disc in which we later plan to fix our base point with an asterisk.



(a) Two discs with a right handed twist

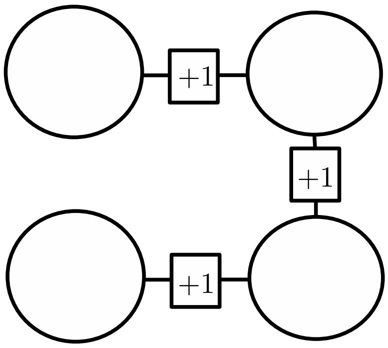


(b) In my notation

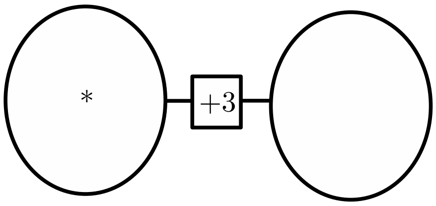
Figure 4.6: Notation for Seifert surfaces

If we can find a chain of discs connected by bands that forms a loop from our base disc and where no disc in the chain is connected to any other disc, then we can clearly replace the chain by a band with multiple twists. In my notation: we replace a chain of circles and lines by a line connecting the base circle back to itself and the number in the box is the sum of all the numbers in the chain. Call this number the *degree* of the line.

We also require a way to reduce the number of discs in the surface so that we are left with only one.



(a) A chain of discs



(b) The simplified diagram

Figure 4.7: Suppressing chains of discs

Consider what happens if we have two discs joined by a half twist. If we then cut down the middle of one disc and along the twist and then treat the two half-discs as discs in their own right, we have the base disc joined to two other discs by bands with a half-twist. But one band will pass over the other, corresponding to the direction of the original twist. (Note also that the two new bands are twisted the same way as the original).

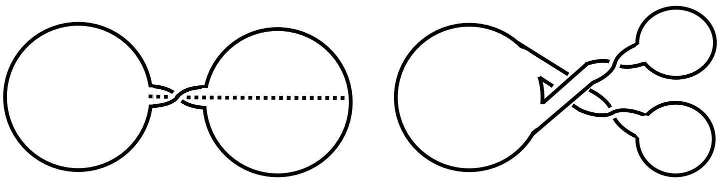


Figure 4.8: Splitting one disc into two

This appears to create more discs but notice that if two discs are joined by three bands, then cutting down the centre of the middle band forms two disc-chains that we can simplify to twisted bands. Also, if two discs are joined by two bands but each has a looped band attached to it, we can cut down the centre of the bands that join the discs together and turn the second disc with its loop into two loops attached to the first disc.

In my notation, we represent this operation by splitting a disc in two and replacing the straight line by two crossed lines with the same degree. Now, as we move out from the base disc, if the order of the original line was -1 then the left line passes over the right. If the order of the original was $+1$ then the right passes over the left. Similarly, for larger orders, the lines cross again in the same way.

Remarks 4.4

This transformation does not change the topological type of the surface. Although it is described as 'cutting' the disc, it can equally be viewed as a continuous deformation by pushing the boundary of the disc in and down the twisted band until it

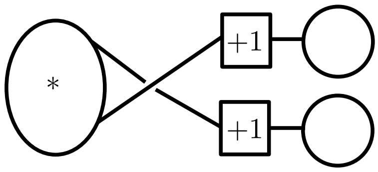


Figure 4.9: Splitting discs with my notation

is level with the boundary of the base disc.

I do not give an algorithm here for creating a simplified Seifert surface but the methods outlined above are sufficient to create simplified surfaces for knots of low crossing numbers without much difficulty. In particular, the previous examples: 3_1 and 5_2.

4.5 Forming the Seifert matrix

We are given a simplified Seifert surface with a base disc (circle) and n bands (lines) radiating from it which twist on themselves and around each other. It is the way in which these bands twist that give us the required information to form a matrix from which we can derive the Alexander polynomial in a different way. Since we began with an orientable surface and the simplifying process has not altered the surface topologically, each band must have an even number of half-twists in it (equivalently: a whole number of full twists). If a band had an odd number of twists, we would be forced to paint the bottom of the base disc the same colour as the top and the surface would be nonorientable.

Label the bands attached to the base disc with symbols $a_1, \ldots, a_n$. We create an $n \times n$ matrix V and label each row and column with $a_1, \ldots, a_n$. The number of full twists of each band determines the leading diagonal of the matrix. ie. If the line representing band a_1 has order $+6$ then the element v_{11} of V will be 3.

The number of times a band crosses another determines the other elements of the matrix. ie. As we move out from the base disc, if band a_1 crosses over band a_2 from left to right m times then the element v_{12} of V will be m. If it crosses from right to left m times then v_{12} will be $-m$.

If a band passes only underneath another band or if bands do not cross each other at all then the corresponding elements of the matrix are zero.

Definition 4.5 *A matrix with its entries filled in in the manner described above is called a* Seifert matrix *for a given Seifert surface F.*

Theorem 4.6 *If V is a Seifert matrix for a Seifert surface of a knot K, then we can obtain its Alexander polynomial by the formula:*

$$\Delta_K(t) = \det(V - tV^T)$$

(V^T denotes the transpose of the matrix V)

The proof of the theorem is omitted.

4.6 The Seifert matrix of 3_1

The Seifert surface we earlier obtained from the diagram of the trefoil consisted of two discs, one above the other, joined together by three bands, each with a right-handed twist. We level the surface to get two discs next to each other, joined again by three bands with right-handed twists. We represent this by two circles, joined by three lines of order $+1$ each.

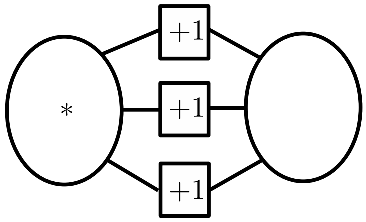


Figure 4.10: The surface for the trefoil

Choose the left-hand circle as our base and split the right-hand circle along the middle band. The middle line is then replaced by two crossed lines of order $+1$ with the right passing over the left. The two circles joined to the base circle can then be suppressed and the orders of the lines added to leave two lines looping back to the base circle of order $+2$ each. Call the top a_1 and the bottom a_2.

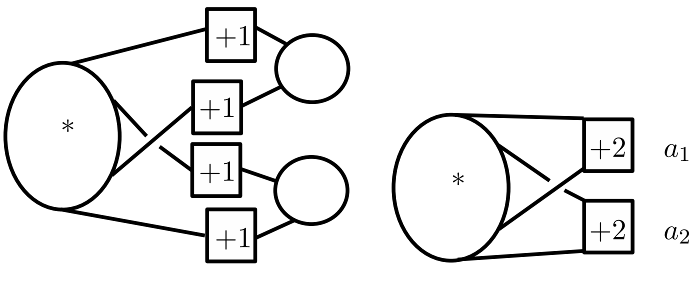


Figure 4.11: Obtaining the simplified diagram

Then a_1 passes over a_2 from right to left and so the matrix element v_{12} is -1. As each band has one full positive loop we have $v_{11} = v_{22} = 1$.
Hence the Seifert matrix is:

$$V = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

We then calculate the Alexander polynomial using the above theorem:

$$\begin{aligned}\Delta_{3_1}(t) = \det(V - tV^T) &= \det\left\{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - t\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right\} \\ &= \begin{vmatrix} 1-t & -1 \\ t & 1-t \end{vmatrix} \\ &= (1-t)^2 + t \\ &= 1 - t + t^2\end{aligned}$$

and we see that we obtain the same polynomial as calculated in Chapter 2.

4.7 The Seifert matrix of 5_2

The Seifert surface we obtain from 5_2 is made up of four discs, which can immediately be simplified to two discs joined by two bands with a right-handed half-twist and one band with three right-handed half-twists. In my notation, we represent this by two circles joined by three lines of order $+1, +1$ and $+3$.

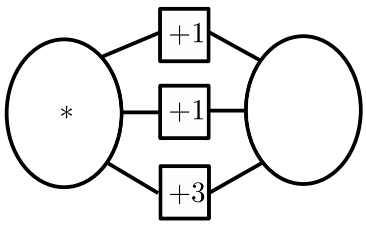


Figure 4.12: The surface for 5_2

In the same way as in the case of the trefoil, we choose the left-hand circle as the base and split the right-hand circle along the middle band. Again, we replace the middle line with crossed lines of order $+1$ with the right crossing over the left. Finally, we suppress the circles to give a line of order $+2$ and a line of order $+4$.

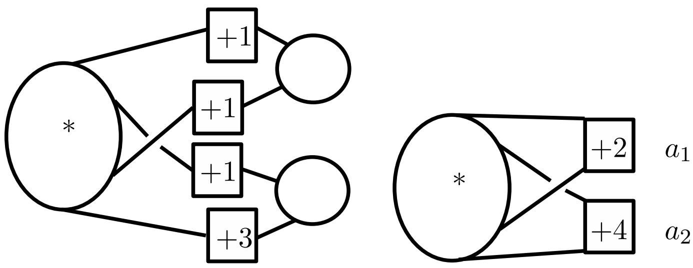


Figure 4.13: Obtaining the simplified diagram

Labelling the bands a_1 and a_2 as before we form the Seifert matrix:

$$V = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

And the Alexander polynomial can similiarly be found by:

$$\begin{aligned}
\Delta_{5_2}(t) = \det(V - tV^T) \quad &= \det \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} - t \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} \right\} \\
&= \begin{vmatrix} 1-t & -1 \\ t & 2-2t \end{vmatrix} \\
&= 2(1-t)^2 + t \\
&= 2 - 3t + 2t^2
\end{aligned}$$

which also agrees with the original calculation.

Chapter 5

The fundamental group: the algebraic route

In Alexander's paper, he also discusses a second invariant of knots: the *group* of the Knot. In this section, instead of assigning a polynomial to each knot, we assign an algebraic structure. These structures form invariants in that whenever two knots have the same type, there exists an isomorphism between their groups.

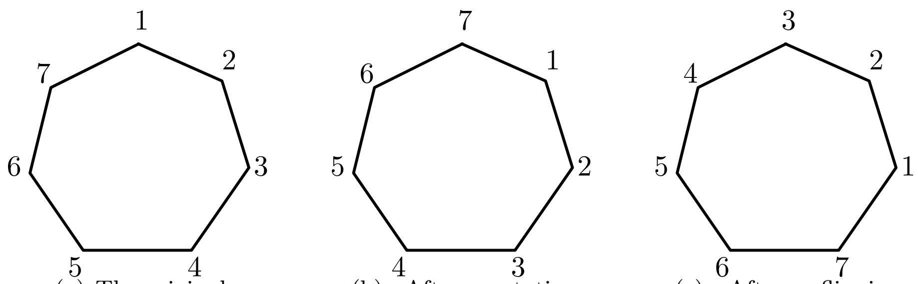
5.1 Abstract groups and group presentations

Informally, a *group* is a collection of *elements* with a single *operation*. The operation is typically addition or multiplication, but transformations of geometric objects can also form a group such as the group of rotations and reflections of a regular polygon. (Note that rotation and reflection are not two different group operations: rather, each is an element of the group and the operation is composition, ie. performing one after the other). A group must also obey certain constraints on its elements and its operation.

Formally:

Definition 5.1 *A* group G *is a pair* $(S, \cdot)$ *of a set together with an binary operation on the elements of the set. G must be closed under this operation and also satisfy the following:*
(1) G is associative*: for all elements* a, b, c*, we require that* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(2) There exists an identity element *denoted by* 1 *such that for all* $a \in G$*,* $1 \cdot a = a \cdot 1 = a$
(3) Each element a must have an inverse*,* a^{-1}*, such that* $a \cdot a^{-1} = a^{-1} \cdot a = 1$

For example, consider a regular heptagon lying in the plane with its corners numbered. We can rotate the heptagon by multiples of $\frac{2\pi}{7}$ or flip it on any of the axes passing through one of its corners and its centre. These transformations form a group. The group identity is leaving the heptagon as it is (we can think of this as a rotation by zero). Clockwise rotations have anticlockwise rotations as inverses and all flips are self-inverse. That is, if you perform them twice then you return to your original position.



(a) The original

(b) After rotation clockwise by $\frac{2\pi}{7}$

(c) After flipping on the axis passing through corner 2

Figure 5.1: Transformations of a heptagon

In the case of *abstract groups*, we do not require the elements of the group to represent 'things' in the real world (such as numbers, rotations, functions etc.) and instead simply use formal symbols. The operation on the symbols is concatenation to form *words*. For example, the combination of symbols $a \cdot b \cdot a^{-1} \cdot c$ forms the word $aba^{-1}c$.

A *presentation* of an abstract group is denoted by:

$$\langle a_1, a_2, \ldots, a_n \rangle$$

which represents the group of all words formed by the symbols $a_1, a_2, \ldots, a_n$ and their inverses. The symbols a_i are called the *generators* of the group. We can also impose a further structure on the group with a presentation:

$$\langle a_1, a_2, \ldots, a_n \mid c_1(a), c_2(a), \ldots, c_m(a) \rangle$$

where each c_i is an identity of the form:

$$a_{i_1}^{\pm\lambda_1} a_{i_2}^{\pm\lambda_2} \cdots a_{i_N}^{\pm\lambda_N}$$

The identities c_i are referred to as the *relations* of the group and denote words which are equal to the identity.

For example, the group presentation $\langle a, b, c \mid a^2, b^2, c^2, abc \rangle$ represents a group with three generators. All elements of the group are words in a, b, c and their inverses and $a^2 = b^2 = c^2 = abc = 1$.

5.2 Application to knots

In order to find a group presentation associated to a knot, we need to find a way of encoding the geometric properties of a knot in an algebraic structure. We do this

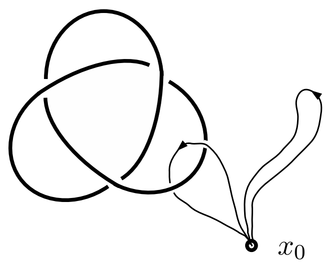


Figure 5.2: Paths in $\mathbb{R}^3 - K$

by considering loops in the space $\mathbb{R}^3 - K$: the complement of the knot.

Fix a base point x_0 somewhere in $\mathbb{R}^3 - K$ and consider the collection of all paths that begin and end at that point. We can define a composition of two paths by travelling down the first path and then down the second.

We consider two paths to be equivalent if one may be continuously deformed into the other within $\mathbb{R}^3 - K$ (ie. without passing through the knot). In the diagram, p_1 and p_1' are equivalent, as are p_2 and p_2'. This means that any path that does not pass around an arc of the knot can be shrunk down to the base point (a constant path). In the diagram, loops p_2 and p_2' are both equivalent to the constant path. Formally, the continuous function which maps one path onto an equivalent path is called a *homotopy* of paths. Call the equivalence class of all loops equivalent to a particular loop p a *loop class.* The composition of two loop classes is well-defined and is taken to be the class of the composition of the loops.

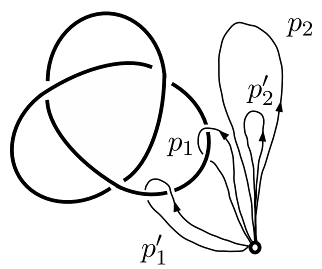


Figure 5.3: Equivalent paths

If we consider the collection of all loop classes in our space then we see that there arises a natural inverse for each class. If a composite loop is formed by travelling down a loop in one direction and then returning to the base point down the same path but in the other direction then we may continuously deform the composite path to the base point. If the original loop was p, call the same loop with the opposite orientation $\bar{p}$. Then the composite loop $p\bar{p}$ lies in the loop class of the constant path at the base point.

Theorem 5.2 *The collection of all loop classes in* $\mathbb{R}^3 - K$ *forms a group with composition of loop classes as the group operation.*

We have seen that each class has an inverse. We take the class of the constant path at the base point to be the group identity. To make these concepts precise it is necessary to define explicit homotopies between paths in the space. Using these homotopies we can also show that composition of loop classes is an associative operation, but I shall omit it here. A full treatment can be found in any introductory text on algebraic topology.

5.3 Generators and relations of the fundamental group

We can read off a set of generators for the knot group from the diagram of the knot. Indeed, each region corresponds to a generator of the group. By convention, we place the base point for the loops in the outside region r_0 and so this element is the group identity. We define the loops corresponding to each region as starting from some point in r_0, passing 'above' the knot diagram, through some region r_i and back 'underneath' the diagram (or the loops can be thought of as originating from the eye of the reader, passing through the region of the knot and returning from underneath the page). For simplicity, we represent the group element by the same symbol as the label of the region.

But what if a loop passes through a number of regions? Can we verify that it is equivalent to a word in the generators $r_0, \ldots, r_{v+1}$? This is easy to show. If a loop r_* passes through region r_1 from top to bottom, then through region r_2 from bottom to top and finally through region r_3 from top to bottom before returning to the base point, we can imagine continuously deforming the loop so that it visits the base point again in between each region (shown by the dotted lines in the diagram). Hence r_* is in fact equivalent to the word $r_1 r_2^{-1} r_3$. Clearly, this argument can be applied to any possible loop around the knot and the reader is invited to try a number of examples to verify the fact.

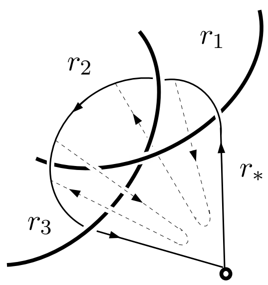


Figure 5.4: A compound loop through three regions

We also need to define a number of relations to give a full presentation of the knot group. Firstly, we must denote the outside region as the identity of the group, since any loop staying in that region can be shrunk to the base point. To show this we place the relation r_0 in the presentation of the knot group. Now consider a crossing point of the diagram with surrounding regions r_j, r_k, r_l, r_m in the same cyclic order as outlined in Chapter 2. For each such crossing point, we also obtain the identity[1]:

$$r_j r_k^{-1} r_l r_m^{-1} = 1$$

and so we add the relation $r_j r_k^{-1} r_l r_m^{-1}$ to the group presentation.

We see the reasoning for this relation by attempting to draw the loop it represents. The loop passes below the overpass at the crossing and above the underpass. Hence the entire loop can be pulled free to lie outside the knot. So compound loops of this type are all equal to the identity.

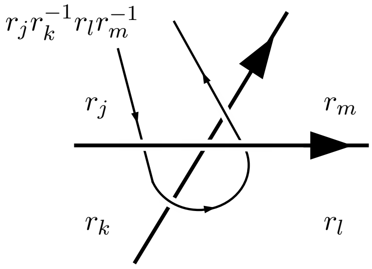


Figure 5.5: A loop that can be pulled free from the knot

Denote these identities by $c_i(r)$, analogously to the equations at the crossing points in Chapter 2 and we have a group presentation for a knot K with v crossing points and $v + 1$ regions:

$$G(K) = \langle r_0, r_1, \ldots, r_{v+1} \mid r_0, c_1(r), \ldots, c_v(r) \rangle$$

5.4 Presentations of $G(3_1)$ and $G(5_2)$

Recall that the defining equations of 3_1 from Chapter 2 are:

$$c_1(r) = tr_0 - tr_3 + r_4 - r_1 = 0$$

$$c_2(r) = tr_0 - tr_1 + r_4 - r_2 = 0$$

$$c_3(r) = tr_0 - tr_2 + r_4 - r_3 = 0$$

From these, we can read off the appropriate relations:

$$c_1(r) = r_0 r_3^{-1} r_4 r_1^{-1}$$

$$c_2(r) = r_0 r_1^{-1} r_4 r_2^{-1}$$

$$c_3(r) = r_0 r_2^{-1} r_4 r_3^{-1}$$

But since r_0 is the identity, we can suppress all instances of it and its inverse. Hence the presentation of the group of the trefoil is:

$$G(3_1) = \langle r_0, r_1, r_2, r_3, r_4 \mid r_0, r_3^{-1} r_4 r_1^{-1}, r_1^{-1} r_4 r_2^{-1}, r_2^{-1} r_4 r_3^{-1} \rangle$$

[1]Alexander uses additive notation in his paper but I shall use multiplicative notation to stress the fact that the knot group is noncommutative

In the case of 5_2 we obtain the relations:

$$c_1(r) = r_1 r_2^{-1} r_3 r_0^{-1}$$

$$c_2(r) = r_1 r_4^{-1} r_3 r_2^{-1}$$
$$c_3(r) = r_1 r_0^{-1} r_5 r_4^{-1}$$
$$c_4(r) = r_6 r_0^{-1} r_3 r_4^{-1}$$
$$c_5(r) = r_6 r_4^{-1} r_5 r_0^{-1}$$

which, when simplified, give the group presentation:

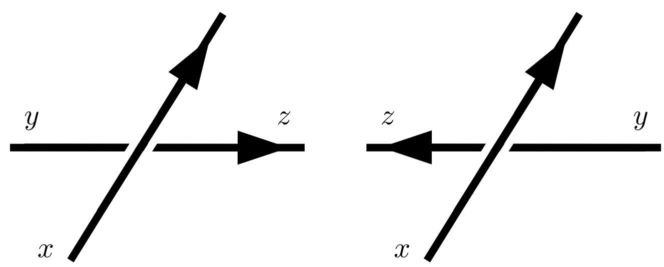
$$G(5_2) = \langle r_0, r_1, r_2, r_3, r_4, r_5, r_6 \mid r_0, r_1 r_2^{-1} r_3, r_1 r_4^{-1} r_3 r_2^{-1}, r_1 r_5 r_4^{-1}, r_6 r_3 r_4^{-1}, r_6 r_4^{-1} r_5 \rangle$$

Chapter 6

Labellings of diagrams

An alternative algebraic treatment of knots is to view the diagram as a collection of separate arcs, breaking where they pass underneath a crossing, and to label the arcs of the diagram with elements from a group.

Definition 6.1 *A* labelling *of an oriented knot diagram D with elements from a group G consists of assigning an element of the group to each arc of the diagram subject to the following conditions:*
(1) Consistency: suppose the overpass at a crossing is labelled with element x, the underpass before the crossing is labelled with element y and the underpass after the crossing is labelled with element z.



(a) Left-handed crossing (b) Right-handed crossing

Figure 6.1: Labelled crossing points

Then the group elements must satisfy:

$$xzx^{-1} = y \text{ at a right-handed crossing}$$
$$\text{and } xyx^{-1} = z \text{ at a left-handed crossing.}$$

(2) Generation: the collection of labels used in the diagram must generate the whole group.

A simple example of a group with which we can try to label knot diagrams is S_n: the group of permutations on n elements. For example, S_3 is the group of all permutations of three elements:

$$s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad s_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$s_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad s_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad s_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

where the permutation sends the top number in the bracket to the number underneath it. Notice that s_1 is the identity permutation.
Permutations are combined by applying one permutation after another. For example:

$$s_2 \cdot s_3 : \begin{cases} 1 \to 2 \to 1 \\ 2 \to 1 \to 3 \\ 3 \to 3 \to 2 \end{cases}, \qquad \text{hence } s_2 \cdot s_3 = s_4$$

6.1 Labelling 3_1 and 5_2 with group elements

It is possible to label 3_1 with elements from the group S_3 as shown. Each element used in the labelling interchanges two of the elements in $\{1, 2, 3\}$. Permutations of this form are called *transpositions*.

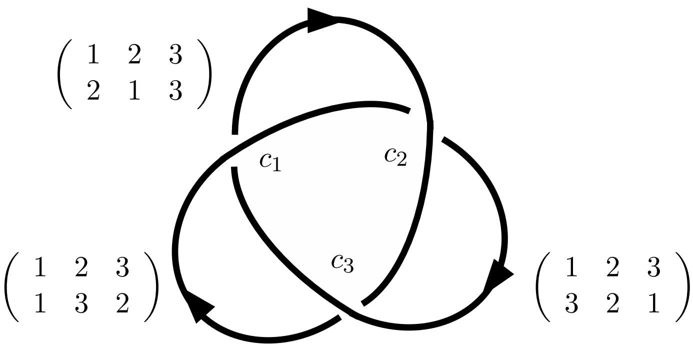


Figure 6.2: A labelling of 3_1 with elements of S_3

We verify the two necessary conditions for the labelling shown to be legitimate:
(1) Consistency: at the three crossing points we check the consistency condition.

$$c_1 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$c_2 : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$c_3 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

(2) We must also show that these elements generate the group. Composing any transposition with itself gives the identity so we only need to find the remaining two group elements in terms of these generators. We find that:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\text{and } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Hence the labels in the diagram generate the whole group and it is a legitimate labelling.

5_2 can be labelled with elements from the group of permutations on seven elements: S_7. For this example it will be convenient to introduce a different form of notation for our permutations. To denote a transposition, we write the two transposed numbers between parentheses. For example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 4 & 5 & 6 & 3 \end{pmatrix} \text{ can be written more simply as } (37)$$

The group elements we use to label the arcs of 5_2 are all products of transpositions. We denote products of transpositions by writing a string of parentheses of the form above. So an element which switches 1 with 6, 2 with 5 and 3 with 4 is denoted (16)(25)(34). The labelling of 5_2 is shown below:

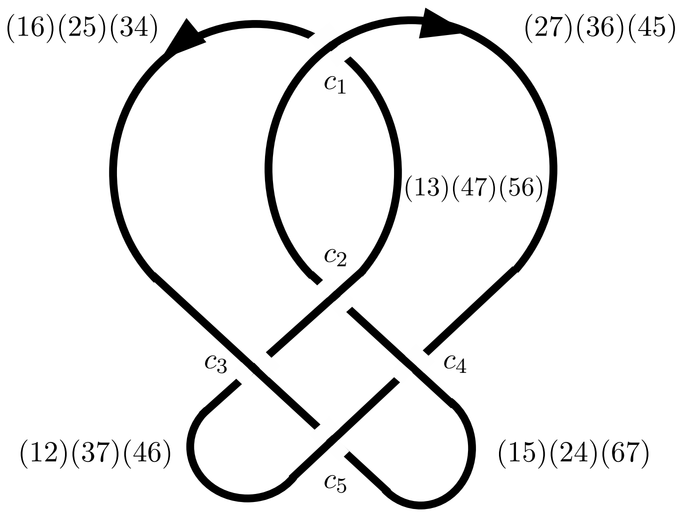


Figure 6.3: A labelling for 5_2

It remains to verify that these elements satisfy the conditions of consistency and generation. Recall that a transposition is self-inverse. If a product of transpositions involves distinct elements then it is also self-inverse. This observation will help in verifying consistency.

(1) Checking all crossing points (see diagram):

$$c_1 : (27)(36)(45) \cdot (16)(25)(34) \cdot (27)(36)(45) = (13)(47)(56)$$

$$c_2 : (13)(47)(56) \cdot (27)(36)(45) \cdot (13)(47)(56) = (15)(24)(67)$$

$$c_3 : (16)(25)(34) \cdot (13)(47)(56) \cdot (16)(25)(34) = (12)(37)(46)$$

$$c_4 : (15)(24)(67) \cdot (12)(27)(46) \cdot (15)(24)(67) = (27)(36)(45)$$

$$c_5 : (12)(37)(46) \cdot (15)(24)(67) \cdot (12)(37)(46) = (16)(25)(34)$$

(2) The elements of S_7 chosen in this labelling do not generate the entire group, in fact they generate a subgroup of S_7 that we looked at in the previous chapter: the group of transformations of a regular heptagon. Each group element describes flipping the heptagon along a particular axis. For example, the element (13)(47)(56) describes the flip along the axis that passes through corner number 2 of the heptagon, shown in the figure in Chapter 5. Similarly, the other group elements are flips along axes which pass through corners 1,3,5 and 7.

The composition $(12)(37)(46) \cdot (27)(36)(45)$ gives us the permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

This is a clockwise rotation of the heptagon by $\frac{2\pi}{7}$. Composing this rotation with itself gives us all the rotations of the group. The only remaining group elements we need now are the flips in axes passing through corners 4 and 6. For corner 4, we can do this by first rotating the heptagon by $\frac{6\pi}{7}$ to put corner 1 in 4's position, then reflecting in the axis through corner 1 and finally using the inverse rotation to return 4 to its starting place. The process is analogous for the flip at corner 6. Hence this is also a legitimate knot labelling.

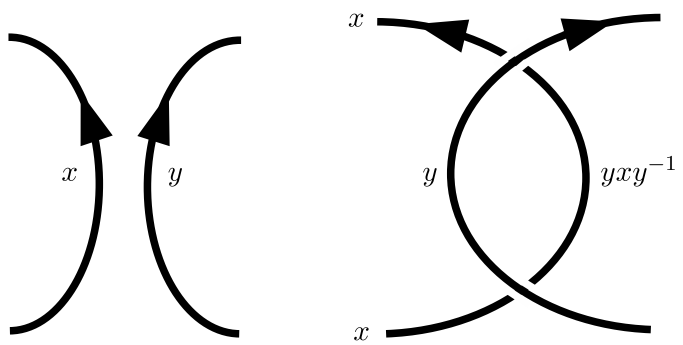
6.2 Invariant properties of labellings

These diagram labellings also form a knot invariant.

Theorem 6.2 *If a labelling exists for a diagram D_1 of a knot K, then a labelling exists with elements of the same group for any other diagram D_2 of K.*

The proof of this involves performing Reidemeister moves on a labelled diagram and seeing whether the labelling can be reconstructed with elements from the same group after the move. Because the consistency condition varies depending on whether a crossing is right or left handed, there are a fairly large number of cases to consider. We will examine two of these as the treatment of the other cases is similar.

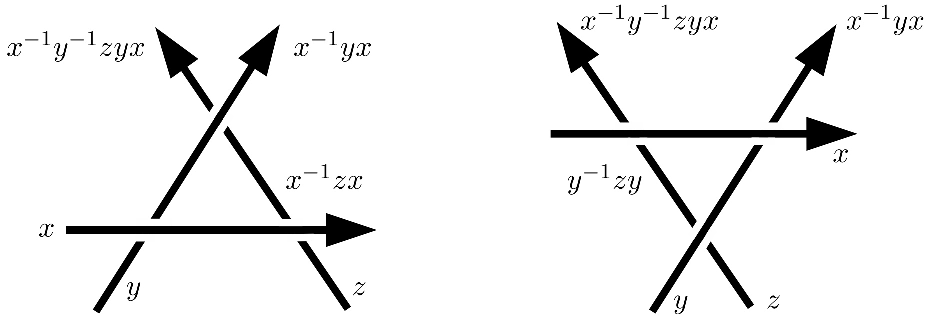
First, consider a Reidemeister II move in which both arcs of the diagram are oriented from the bottom of the diagram to the top. The left arc is labelled with element x and the right arc is labelled with element y. After the move, a left-handed crossing point and a right-handed crossing point are formed. Both crossing points force the same labelling on the new arc by the consistency condition. We label it with the element yxy^{-1}. Note that if the original labelling generated the group then so does the new labelling since x and y still appear and no new generators are included.



(a) Before move (b) After move

Figure 6.4: Reidemeister II move with arc labels

For another case, consider a Reidemeister III move with the arcs oriented as shown in the diagram. Labelling the arcs x, y and z as shown forces the labelling of the remaining arcs of the diagram. The arc leaving the top left of the diagram has label $x^{-1}y^{-1}zyx$ and the arc leaving the top right has label $x^{-1}yx$. Now, if we label the same arcs with x, y and z after the Reidemeister move, we see that there is a new arc in the middle of the diagram with label $y^{-1}zy$ but the arcs leaving the top left and top right of the diagram end up with the same labels as before. Again, we use the same set of generators so this new labelling satisfies both consistency and generation and the labels come from the same group.



(a) Before move (b) After move

Figure 6.5: Reidemeister III move with arc labels

6.3 Relation to presentation of the knot group

We can use the conditions for a labelling of a knot diagram to give an alternative presentation of the knot group from that given in Chapter 5. Again, treat the group elements which label the diagram as formal symbols and assign one symbol to each arc of the knot diagram. Then, treating these symbols as the generators of the group, clearly the generation condition of the labelling holds. We enforce the consistency condition by using the equations at the crossing points to form the relations of the group presentation.

For example, if a relation at a crossing is:

$$xyx^{-1} = z$$

then we insert the relation $xyx^{-1}z^{-1}$ in the presentation of the group.

In fact, it is simple to demonstrate that these two group presentations lead to isomorphic groups.

Consider a right-handed crossing point. The regions surrounding the point are r_j, r_k, r_l and r_m. The overpass is labelled with symbol x and on the underpass, symbol y labels the back end and symbol z labels the front end.

Then, consider x representing a loop, as described in Chapter 5, which passes around the arc it labels. Note that two such loops are possible: one in each direction. We will take x to be the loop that goes clockwise around the arc as you look along the direction of orientation. (To help visualise: if the right hand grips the arc with the thumb pointing in the direction of orientation then the loop x is oriented in the same sense as the direction the fingers curl around). Similarly define loops for y and z.

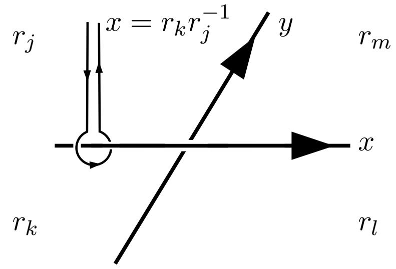


Figure 6.6: A loop around an arc of the diagram

In our diagram, we see that x can be expressed in terms of r_j and r_k. ie. $x = r_k r_j^{-1}$. But note that the relation corresponding to this crossing point gives us an alternative expression:

$$r_j r_k^{-1} r_l r_m^{-1} = 1 \Leftrightarrow r_k r_j^{-1} = r_l r_m^{-1}$$

Hence x can also be expressed as $r_l r_m^{-1}$. Similarly, $y = r_l r_k^{-1}$ and $z = r_m r_j^{-1}$. With these new definitions, we see that the consistency relation still holds since:

$$xzx^{-1} = (r_l r_m^{-1})(r_m r_j^{-1})(r_j r_k^{-1}) = r_l r_k^{-1} = y$$

At a left-handed crossing, the x arc goes in the other direction and so, to preserve the orientation of the loops, we have that $x = r_j r_k^{-1} = r_m r_l^{-1}$.

The expressions for y and z are the same as in the right-handed case. In this case, then, the consistency relation is:

$$xyx^{-1} = (r_m r_l^{-1})(r_l r_k^{-1})(r_k r_j^{-1}) = r_m r_j^{-1} = z$$

as required.

So considering the knot group as generated by loops around the arcs instead of loops through the regions provides a different generating set but with equivalent relations to the original presentation. Note that the relation r_0 does not appear in this second presentation as the identity element is the symbolic identity, 1, which is not an arc label. When using the regions of the diagram as generators for the group, the group presentation will have $v + 2$ generators and $v + 1$ relations (one for each crossing point and one to define the identity element). When we use arc labellings, the group presentation will have v generators and v relations. So using labellings leads to a simpler presentation of the knot group than using the original definition. We will see that the presentation of the group using labellings can also be further simplified.

Note that in the case of a group presentation generated by labellings, one of the group relations can always be deduced from the others (or if there are only two group relations then they are equivalent). Hence we can always omit one of the relations without losing any information about the group. This means that a presentation from a labelling will really have v generators and $v - 1$ relations. So in both cases there is one more generator than relation. We state this by saying the group has *defect* 1, and this is in fact a property of all knot groups.

6.4 The groups of 3_1 and 5_2 generated by labellings

In the case of the trefoil, we label the three arcs of the diagram with symbols x, y and z.

Then the three crossing points require that $zxz^{-1} = y$, $xyx^{-1} = z$ and $yzy^{-1} = x$. Hence the presentation of the group with respect to this particular labelling is:

$$G(3_1) = \langle x, y, z \mid zxz^{-1}y^{-1}, xyx^{-1}z^{-1}, yzy^{-1}x^{-1} \rangle$$

In the case of 5_2, the diagram is broken into five arcs which we label u, v, x, y, z.

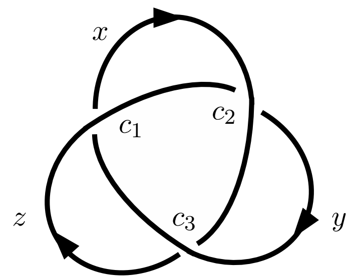


Figure 6.7: Labelling the arcs of 3_1

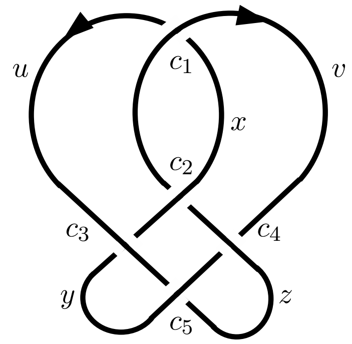


Figure 6.8: Labelling the arcs of 5_2

At crossing c_1 the consistency condition requires that $vuv^{-1} = x$. Hence we obtain the relation $vuv^{-1}x^{-1}$ in the group presentation. By considering the condition at the remaining crossing points we see that the group presentation is:

$$G(5_2) = \langle u, v, x, y, z \mid vuv^{-1}x^{-1}, xvx^{-1}z^{-1}, uxu^{-1}y^{-1}, zyz^{-1}v^{-1}, yzy^{-1}u^{-1} \rangle$$

6.5 Determining a labelling with fewer generators

It is not necessary to use a generator for each arc of the diagram. Indeed, if we start with a smaller number of generators then we can begin at one point in the diagram and use the consistency condition to determine the labels of the arcs each time we come to a crossing point.

For example, in the case of the trefoil labelling two of the arcs with symbols x and y forces the third arc to be labelled with xyx^{-1}.

We then simply require the consistency condition to be met at the remaining two crossing points. ie.

$$c_2 : (xyx^{-1})(x)(xy^{-1}x^{-1}) = y$$
$$c_2 : (y)(xyx^{-1})(y^{-1}) = x$$

So our group representation is:

$$G(3_1) = \langle x, y \mid xyxy^{-1}x^{-1}y^{-1}, yxyx^{-1}y^{-1}x^{-1} \rangle$$

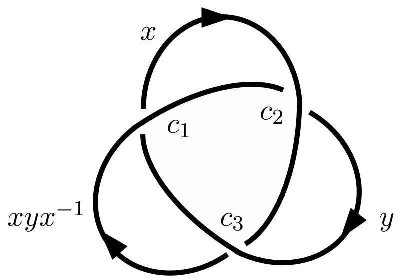


Figure 6.9: Determining a labelling of 3_1 with generators x,y

It has only two generators and two relations and so is a simpler presentation of the group of 3_1.

In the case of 5_2, if we label the top two arcs of the diagram with symbols x and y then the consistency condition at crossing points c_1, c_2 and c_3 determines the labelling of all the remaining arcs in the diagram as shown.

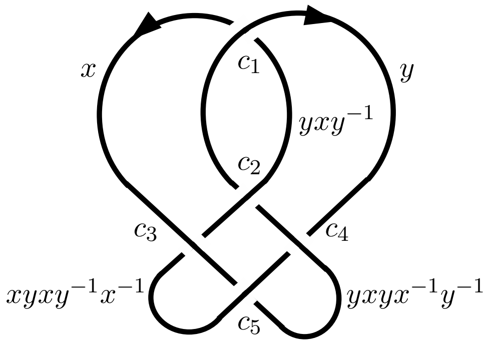


Figure 6.10: Determining a labelling of 5_2 with generators x,y

Once all arcs have been labelled, we obtain the relations for our group presentation from crossing points c_4 and c_5 which require:

$$c_4 : (yxyx^{-1}y^{-1})(xyxy^{-1}x^{-1})(yxy^{-1}x^{-1}y^{-1}) = y$$

$$c_5 : (xyxy^{-1}x^{-1})(yxyx^{-1}y^{-1})(xyx^{-1}y^{-1}x^{-1}) = x$$

These lead to the group presentation:

$$G(5_2) = \left\langle x, y \;\middle|\; \begin{array}{l} xyx^{-1}y^{-1}xyxy^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}, \\ yxy^{-1}x^{-1}yxyx^{-1}y^{-1}xyx^{-1}y^{-1}x^{-1} \end{array} \right\rangle$$

Chapter 7

The Fox algorithm

In this final chapter, we see how the presentation of a knot group can be related back to the central theme of this document, namely the Alexander polynomial. This is via a process developed by the American mathematician RH Fox in which the Alexander polynomial can be recovered from a presentation of a knot group using a form of calculus called *Fox derivatives*.

7.1 Fox derivatives

Fox derivatives are a means of defining a partial derivative for a monomial in noncommuting symbols, such as the words which form the relations in our group presentation. It should be noted that the derivative formed in this way cannot be treated as a word in the same sense as before, but will be a formal sum of words in the same symbols.

Definition 7.1 *Suppose w_1 and w_2 are words in symbols $x_1, \ldots, x_i, x_j, \ldots, x_n$ and their inverses. Then we find the* Fox derivative *of the words using the following rules:*
(1) $\frac{\partial}{\partial x_i}(x_i) = 1$
(2) $\frac{\partial}{\partial x_i}(x_j) = 0$ *if* $i \neq j$
(3) $\frac{\partial}{\partial x_i}(1) = 0$
(4) $\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$
(5) $\frac{\partial}{\partial x_i}(w_1 \cdot w_2) = \frac{\partial}{\partial x_i}(w_1) + w_1 \cdot \frac{\partial}{\partial x_i}(w_2)$

The general method is to differentiate the left-hand symbol at each stage and use rule (5) to obtain a sum of terms until the end of the word is reached.

For example, consider the word $xyxy^{-1}x^{-1}y^{-1}$. Then, applying rules (1) and (5) we see that:

$$\frac{\partial}{\partial x}(xyxy^{-1}x^{-1}y^{-1}) = 1 + x \cdot \frac{\partial}{\partial x}(yxy^{-1}x^{-1}y^{-1})$$

Applying rules (2) and (5):

$$\frac{\partial}{\partial x}(yxy^{-1}x^{-1}y^{-1}) = 0 + y \cdot \frac{\partial}{\partial x}(xy^{-1}x^{-1}y^{-1})$$

Applying (1) and (5) again, twice:

$$\frac{\partial}{\partial x}(xy^{-1}x^{-1}y^{-1}) = 1 + x \cdot (0 + y^{-1} \cdot \frac{\partial}{\partial x}(x^{-1}y^{-1}))$$

Finally, rules (4) and (5) give:

$$\frac{\partial}{\partial x}(x^{-1}y^{-1}) = -x^{-1}$$

Combining the above results we find the Fox derivative of the word:

$$\begin{aligned} \tfrac{\partial}{\partial x}(xyxy^{-1}x^{-1}y^{-1}) &= 1 + x(0 + y(1 + x(0 + y^{-1}(-x^{-1})))) \\ &= 1 + xy - xyxy^{-1}x^{-1} \end{aligned}$$

With a little practice, Fox derivatives can be calculated relatively easily and, in a similar manner, we find that the derivative of the above word with respect to y is $x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}$.

7.2 Obtaining the Alexander polynomial

Given a presentation of a knot group, we will have n words acting as the relations of the group in n variables. Then, using Fox's calculus we may calculate the derivative of each word with respect to each variable in turn and represent their values in an $n \times n$ matrix, called the *Jacobian* of the group presentation:

$$J = \left(\frac{\partial w_i}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} & \cdots & \frac{\partial w_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial w_n}{\partial x_1} & \frac{\partial w_n}{\partial x_2} & \cdots & \frac{\partial w_n}{\partial x_n} \end{pmatrix}$$

We then delete any one row and any one column from the matrix and substitute t for all the variables. For example, for the derivative calculated above we substitute $1 + xy - xyxy^{-1}x^{-1}$ with $1 + t^2 - t^3 \ldots t^{-1}$, which simplifies to $1 - t + t^2$. (Recall that this is the Alexander polynomial for the trefoil: this same calculation will be used in the example later in the chapter).

The matrix we are left with is equivalent to the Alexander matrix of Chapter 2 and hence taking its determinant and normalising the resulting polynomial as before gives us back the Alexander polynomial. I shall not prove this result.

7.3 The Alexander polynomial of 3_1 using Fox derivatives

We use the presentation of the group using arc labellings as discussed in Chapter 6. The trefoil has three arcs which we label x, y and z. Recall that the consistency condition at the crossing points of the diagram gives us the relations:

$$w_1 = zxz^{-1}y^{-1}$$

$$w_2 = xyx^{-1}z^{-1}$$
$$w_3 = yzy^{-1}x^{-1}$$

Then, using the Fox calculus we find that $\frac{\partial w_1}{\partial x} = z$, $\frac{\partial w_2}{\partial x} = 1 - xyx^{-1}$ and $\frac{\partial w_3}{\partial x} = -yzy^{-1}x^{-1}$ By similar calculations for w_2 and w_3 we find the Jacobian matrix:

$$J(3_1) = \begin{pmatrix} z & 1 - xyx^{-1} & -yzy^{-1}x^{-1} \\ -zxz^{-1}y^{-1} & x & 1 - yzy^{-1} \\ 1 - zxz^{-1} & -xyx^{-1}z^{-1} & y \end{pmatrix}$$

Deleting the last row and column and substituting t for x, y and z we get:

$$M = \begin{pmatrix} t & 1-t \\ -1 & t \end{pmatrix}$$

And then clearly $\det(M) = 1 - t + t^2 = \Delta_{3_1}(t)$ as required.

7.4 The Alexander polynomial of 5_2 using Fox derivatives

From Chapter 6, the relations in the group presentation of 5_2 gives us the words $w_1, \ldots, w_5$ in variables u, v, w, x, y and z:

$$w_1 = vuv^{-1}x^{-1}$$
$$w_2 = xvx^{-1}z^{-1}$$
$$w_3 = uxu^{-1}y^{-1}$$
$$w_4 = zyz^{-1}v^{-1},$$
$$w_5 = yzy^{-1}u^{-1}$$

Then the Fox derivatives of w_1 are:

$$\frac{\partial w_1}{\partial u} = v, \quad \frac{\partial w_1}{\partial v} = 1 - vuv^{-1}, \quad \frac{\partial w_1}{\partial x} = -vuv^{-1}x^{-1}, \quad \frac{\partial w_1}{\partial y} = \frac{\partial w_1}{\partial z} = 0$$

A series of similar calculations for the words $w_2, \ldots, w_5$ gives the Jacobian matrix:

$$J(5_2) = \begin{pmatrix} v & 1 - vuv^{-1} & -vuv^{-1}x^{-1} & 0 & 0 \\ 0 & x & 1 - xvx^{-1} & 0 & -xvx^{-1}z^{-1} \\ 1 - uxu^{-1} & 0 & u & -uxu^{-1}y^{-1} & 0 \\ 0 & -zyz^{-1}v^{-1} & 0 & z & 1 - zyz^{-1} \\ -yzy^{-1}u^{-1} & 0 & 0 & 1 - yzy^{-1} & y \end{pmatrix}$$

We delete the final row and column and map all variables to t to get the matrix:

$$N = \begin{pmatrix} t & 1-t & -1 & 0 \\ 0 & t & 1-t & 0 \\ 1-t & 0 & t & -1 \\ 0 & -1 & 0 & t \end{pmatrix}$$

Finally:

$$\begin{aligned}
\det(N) &= \begin{vmatrix} t & 1-t & -1 & 0 \\ 0 & t & 1-t & 0 \\ 1-t & 0 & t & -1 \\ 0 & -1 & 0 & t \end{vmatrix} \\
&= \begin{vmatrix} 0 & 1 & 0 & -t \\ 0 & t & 1-t & 0 \\ 1-t & 0 & t & -1 \\ t & 1-t & -1 & 0 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & 1-t & 0 \\ 1-t & t & -1 \\ t & -1 & 0 \end{vmatrix} + t \begin{vmatrix} 0 & t & 1-t \\ 1-t & 0 & t \\ t & 1-t & -1 \end{vmatrix} \\
&= (1-t)\begin{vmatrix} 1-t & -1 \\ t & 0 \end{vmatrix} - t^2 \begin{vmatrix} 1-t & t \\ t & -1 \end{vmatrix} + t(1-t)\begin{vmatrix} 1-t & 0 \\ t & 1-t \end{vmatrix} \\
&= t(1-t) - t^2\left[(t-1) - t^2\right] + t(1-t)^3 \\
&= 2t - 3t^2 + 2t^3
\end{aligned}$$

Which normalises to $2 - 3t + 2t^2$ giving the required Alexander polynomial for 5_2.

7.5 Using Fox derivatives on the alternative group presentation

Recall that in Chapter 6 we created an alternative group presentation for both 3_1 and 5_2 that had only two generators in each case. The advantage of using the Fox calculus on these presentations is that the Jacobian matrix is only a 2×2 matrix and so when one row and column has been struck out there is only one element left. This basically means that the Fox derivative of either of the relations in the group presentation, with respect to either generator will yield the Alexander polynomial.

In the presentation of the group of the trefoil, the two words in the list of relations are:

$$w_1 = xyxy^{-1}x^{-1}y^{-1}$$
$$w_2 = yxyx^{-1}y^{-1}x{-}1$$

Then, as calculated in our earlier example:

$$\frac{\partial w_1}{\partial x} = 1 + x(0 + y(1 + x(0 + y^{-1}(-x^{-1})))) = 1 + xy - xyxy^{-1}x^{-1}$$

Mapping all symbols to t we are left with $1 - t + t^2$ as required. Similarly:

$$\frac{\partial w_2}{\partial y} = 1 + y(0 + x(1 + y(0 + x^{-1}(-y^{-1})))) = 1 + yx - yxyx^{-1}y^{-1}$$

Which again maps to $1-t+t^2$. In the same way, it can be shown that the remaining two Fox derivatives also yield the Alexander polynomial.

For the group of 5_2 our two words in x and y are:

$$w_1 = xyx^{-1}y^{-1}xyxy^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}$$

$$w_2 = yxy^{-1}x^{-1}yxyx^{-1}y^{-1}xyx^{-1}y^{-1}x^{-1}$$

Then the derivative of w_1 with respect to x is:

$$\begin{aligned}\tfrac{\partial w_1}{\partial x} &= 1 + x(y(-x^{-1} + x^{-1}(y^{-1}(1 + x(y(1+ \\ &\qquad +x(y^{-1}(-x^{-1} + x^{-1}(y(1 + x(y^{-1}(-x^{-1}))))))))))) \\ &= 1 - xyx^{-1} + xyx^{-1}y^{-1} + xyx^{-1}y^{-1}xy - xyx^{-1}y^{-1}xyxy^{-1}x^{-1} \\ &\qquad +xyx^{-1}y^{-1}xyxy^{-1}x^{-1}y - xyx^{-1}y^{-1}xyxy^{-1}x^{-1}yxy^{-1}x^{-1}\end{aligned}$$

Mapping all symbols to t we get:

$$1 - t + 1 + t^2 - t + t^2 - t = 2 - 3t + 2t^2$$

as required.

So in choosing which group presentation to use, we must consider the payoff between the more complicated algebra and the more complicated calculus. If we begin with a larger number of generators then the relations are all very simple to differentiate but we are left with a large matrix for which it takes a considerable effort to find the determinant (if we are not using a computer!). If, on the other hand, we begin with a minimal number of generators, we may not need to calculate determinants at all but the calculation of the Fox derivative is much more cumbersome since the relations of the group are longer words.

Chapter 8

Knot theory redeemed

Alexander's achievements in defining a polynomial invariant for knots paved the way for the development of a number of other polynomial knot invariants including the *HOMFLY polynomial* and the *Jones polynomial*. These are calculated in a different manner from the Alexander polynomial. The calculation of the Jones polynomial, in particular, involves observing the effect on the knot of performing a kind of local surgery by cutting the strands of the knot in a small neighbourhood and reattaching them in a different manner. Altering the knot in this way is called a *skein operation*.

The Jones polynomial was the first polynomial knot invariant to be discovered since the Alexander polynomial and it took until 1984 for this development to take place. The Jones polynomial is considered to be a more powerful knot invariant than the Alexander polynomial as it distinguishes a larger proportion of knot types (it distinguishes all prime knots with fewer than 9 crossings and also recognises the handedness of knots) although it is still not a complete invariant and there even exist pairs of knots that have the same Jones polynomial but different Alexander polynomials.

Also, despite spending the most part of a century relegated to the attention of only a select number of pure mathematicians and enthusiasts, knot theory has recently enjoyed a return to popularity within the wider scientific community. This is because the mathematical tools developed for the study of knot types such as the invariants we have discussed have been found to have uses in studying a number of phenomena in physics, biology and chemistry. Given that the early interest in the study of knots was amongst the physics community it is fitting that finally the mathematics has finally found a practical use again in that field.

8.1 Applications in molecular biology

Every cell in the human (or indeed non-human) body contains all of the genetic information needed to build the entire body. This information is coded on strands of DNA, the famous double-helix structure discovered by James Watson, Francis Crick and Rosalind Franklin in 1953 and for which Watson and Crick were awarded the Nobel Prize for Medicine in 1962.

Topologically, DNA molecules take the form of long strands which can occur as single strands or as coiled double helices. These either lie along an axis or have the ends attached to form a ring.

In the process of DNA replication, the strands of the DNA are cut up by small enzymes that occur inside the cell called *topoisomerases*. The action of these topoisomerases on strands of DNA is similar to the skein operations on knots described above in calculating the Jones polynomial and the product molecules formed after the action of the enzyme can have the form of various knots and links.

The enzymes may act on a molecule of DNA a number of times, forming a series of topologically different objects with each repeated recombination. Biologists can then use knot theoretic techniques to analyse the topological types of a sample of DNA molecules in a cell and then use this information to infer the specific change to the structure of the DNA molecule each time the topoisomerase interacts with it.

8.2 Applications in statistical mechanics

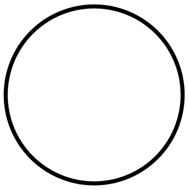
All matter is made up of a vast number of tiny particles which interact with each other in a very complicated way. To understand the macroscopic properties of matter by examining the mechanics of its constituent particles is almost impossible and so what is required is a statistical approach: describing the *probability* that a particle within a larger structure will have a certain property.

Statistical mechanics seeks to model the macroscopic properties of matter using idealised statistical models and considering the respective probabilities that various *states* of the model will be realised.

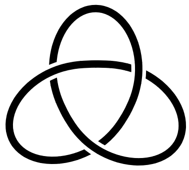
The diagrams used to represent this idealised model of matter are in the form of lattices of intersecting lines, which can be interpreted as arcs of a knot or link and the points of intersection as crossing points.

Finally, the relations between the different states of the model have been found to correspond again to skein operations so the mathematical properties of the Jones polynomial can give insights into the properties of statistical mechanical models.

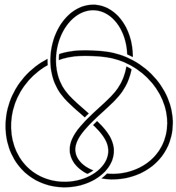
A table of prime knots of 7 or fewer crossings with their Alexander polynomials



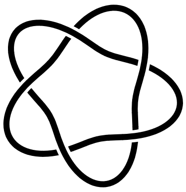
$\mathbf{0_1}$
1



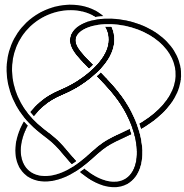
$\mathbf{3_1}$
$1 - t + t^2$



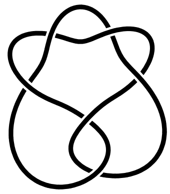
$\mathbf{4_1}$
$1 - 3t + t^2$



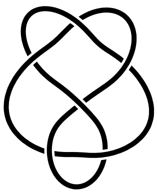
$\mathbf{5_1}$
$1 - t + t^2 - t^3 + t^4$



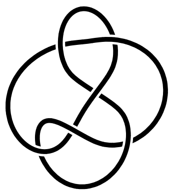
$\mathbf{5_2}$
$2 - 3t + 2t^2$



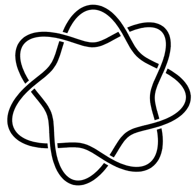
$\mathbf{6_1}$
$2 - 5t + 2t^2$



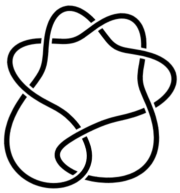
$\mathbf{6_2}$
$1 - 3t + 3t^2 - 3t^3 + t^4$



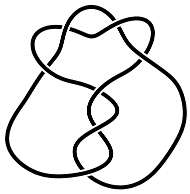
$\mathbf{6_3}$
$1 - 3t + 5t^2 - 3t^3 + t^4$



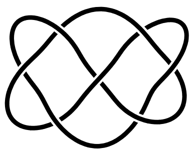
$\mathbf{7_1}$
$1 - t + t^2 - t^3 + t^4 - t^5 + t^6$



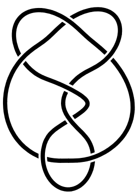
$\mathbf{7_2}$
$3 - 5t + 3t^2$



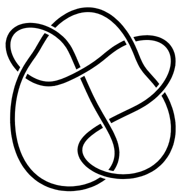
$\mathbf{7_3}$
$2 - 3t + 3t^2 - 3t^3 + 2t^4$



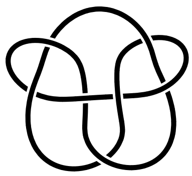
$\mathbf{7_4}$
$4 - 7t + 4t^3$



$\mathbf{7_5}$
$2 - 4t + 5t^2 - 4t^3 + 2t^4$



$\mathbf{7_6}$
$1 - 5t + 7t^2 - 5t^3 + t^4$



$\mathbf{7_7}$
$1 - 5t + 9t^2 - 5t^3 + t^4$

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Notes

Text typeset in LaTeX using TeXshop. The knot diagrams were drawn using the Knotplot program, all hand-drawn diagrams were drawn using Stardraw. The front cover graphics were produced using The GIMP (the knot graphic is from the Knotplot site).