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The Planck constant of action and the Kibble balance

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ABSTRACT

It has been shown previously (P. R. Bunker and Per Jensen, *J. Quant. Spectrosc. Radiat. Transf.*, **243** (2020) 106835) that if we choose angles to have dimension, we have to distinguish between the Planck constant h, having the dimension of $\mathrm{action\,angle}^{-1}$, and the Planck constant of action h_{A}, having the dimension of action. In the present paper, we show that a further implication that results from choosing angles to have dimension is that the Kibble balance equation relating the mass weighed to the Planck constant has to involve both of the distinct fundamental constants h and h_{A}. We derive that new equation here and show how it compares to the equation that is obtained if one chooses angles to be dimensionless as required in SI.

1. Introduction

In the SI, angular measure is considered to be a dimensionless quantity despite the fact that it undoubtedly has units. The advantages that follow from giving angles their own dimension, and treating this dimension in a consistent way, are discussed in the recent paper by Mohr et al. [1], but the matter is controversial. We are concerned with one particular consequence of choosing angular measure to be a dimensioned quantity, namely its effect on the dimension and units of the Planck constant and of equations involving the Planck constant.

It has been shown previously by Bunker and Jensen [2] that if we consider angular measure to be a dimensioned quantity then it is necessary to distinguish between the Planck constant h, having the dimension of $\mathrm{action\,angle}^{-1}$, and the Planck constant of action (called h_{A} in [2]), having the dimension of action. In the present paper, we show that a further consequence is that the Kibble balance equation relating the mass weighed to the Planck constant has to include both of these distinct fundamental constants. We derive that new equation here and contrast it to that which one obtains when, following SI, angles are chosen to be dimensionless (see, for example, Robinson and Schlamminger [3] and Schlamminger [4]). This equation is important for the SI definition of the kilogram [5].

2. The Planck constant and the Planck constant of action

We first summarize the results in Bunker et al. [6] and Bunker and Jensen [2] concerning the values, numerical values and units for the Planck constant and the Planck constant of action when angles are chosen to have their own dimension. Following the rules of quantity calculus [7], we write each of these values as the product of its numerical value and its unit. As shown in Bunker et al. [6], when angles are chosen to have a dimension the Planck constant h and the reduced Planck constant $\hbar$ have the values

$$h = 6.62607015 \times 10^{-34}\ \mathrm{J\ s\ cycle}^{-1} \tag{1}$$

and

$$\hbar = \left[\frac{6.62607015}{(2\pi)}\right] \times 10^{-34}\ \mathrm{J\ s\ radian}^{-1}. \tag{2}$$

The values of h and $\hbar$ are identical, that is

$$h = \hbar, \tag{3}$$

but their numerical values are in the ratio 2π, being a number that is the ratio of the angular units cycle and radian. This equality relation between the values of h and $\hbar$, when angles are chosen to have their own dimension, has been pointed out before [8,9].

If we choose angles to have dimension, the Planck constant appears in the equations of quantum mechanics with the dimension of action, without the involvement of the angular dimension. This is a new fundamental constant, distinct from h which has the dimension of action angle^{-1}. It was introduced by Bunker and Jensen [2] and called the Planck constant of action h_{A}. The values of h_{A}, and of the reduced Planck constant of action $\hbar_{\mathrm{A}}$, both with the dimension of action J s, are given by

$$h_{\mathrm{A}} = 6.62607015 \times 10^{-34}\ \mathrm{J\ s} \tag{4}$$

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and

$$\hbar_{\rm A} = \left[\frac{6.62607015}{(2\pi)}\right] \times 10^{-34} \text{ J s}. \tag{5}$$

The values of $h_{\rm A}$ and $\hbar_{\rm A}$ are in the ratio 2π, as are their numerical values. As pointed out in [2], when angular measure is dimensioned, h (in Eq. (1)) and $h_{\rm A}$ are different constants with different units, but they are intimately related since, to agree with experiment, they have the same numerical values when expressed in the units J s cycle^{-1} and J s, respectively.

In SI, when angular measure is chosen to be dimensionless, h and $\hbar$ are both said to have the units J s and their values (and numerical values) are in the ratio 2π. Also, in SI, the units s^{-1} and Hz are taken to be synonymous, and the units J s can be written J Hz^{-1}.

The equation relating photon energy E to electromagnetic radiation frequency f is

$$E = hf. \tag{6}$$

When angles are chosen to be dimensioned, h has units J s cycle^{-1}, as in Eq. (1), and f has units cycle s^{-1} to give E in J. But, if (following SI) angles are chosen to be dimensionless, then in this equation h would be said to have units J s and f would be said to have unit s^{-1} (or Hz) to give E in J. However, electromagnetic radiation frequency can be given in either radian s^{-1} or cycle s^{-1}. In SI, there is a convention that when frequency is written as f, the value of it in cycle s^{-1} is to be used. See Bunker et al. [6] for a more detailed discussion of this state of affairs when angles are considered to be dimensionless.

3. The Kibble balance equations

In using the Kibble balance to relate a mass to the Planck constant one performs a static experiment and a dynamic experiment. In the static experiment, one determines the current I in a coil of length l within a magnetic field of flux density B that is required in order to balance the magnetic force on the coil against the weight Mg of a mass M. In the dynamic experiment the mass is removed and one measures the voltage V induced in the coil by moving the coil through the magnetic field B at a constant velocity v. The local acceleration due to gravity g is measured using a gravimeter, and the velocity v is measured using an interferometer and an atomic clock. Assuming that B and l are the same in the static and dynamic experiments, one deduces the relation

$$M = VI/(gv). \tag{7}$$

The derivation of the equation that relates VI in watts, and hence the mass M in kilograms, to the Planck constant h is described, for example, in Robinson and Schlamminger [3] and Schlamminger [4] where, to follow SI, angles are chosen to be dimensionless. Before giving that derivation, it is necessary to introduce equations involving the Josephson constant $K_{\rm J}$ and von Klitzing constant $R_{\rm K}$, and to discuss the units in these equations when angles are chosen to be dimensionless or when chosen to be dimensioned.

To express the voltage V in terms of the Planck constant, we make use of the Josephson effect which gives the voltage of a Josephson junction as:

$$f/K_{\rm J} = hf/2e, \tag{8}$$

where, if angles are chosen to be dimensionless, h is said to have units J s, and the microwave frequency f is said to have unit s^{-1} to give $hf/2e$ in J/(A s) = volt. If angles are chosen to be dimensioned, the Josephson equation Eq. (8) still applies but in it h has units J s cycle^{-1} and f explicitly has units cycle s^{-1}. This is similar to the situation in Eq. (6) which also involves the product hf. Inverting the Josephson equation, we obtain the Josephson constant as

$$K_{\rm J} = 2e/h, \tag{9}$$

whether we choose angles to be dimensioned or not; h has the units J s if angles are chosen to be dimensionless but it has units J s cycle^{-1} if angles are chosen to be dimensioned. This means that the Josephson constant has the units s^{-1} volt^{-1} (or, equivalently, Hz volt^{-1}) if angles are chosen to be dimensionless (following SI), and units cycle s^{-1} volt^{-1} if angles are chosen to be dimensioned.

To express I in terms of the Planck constant requires a second use of the Josephson effect, and the use of the quantum Hall effect (as explained in the next paragraph). The quantum Hall effect involves the von Klitzing constant, and when angles are chosen to be dimensionless, this constant is given by

$$R_{\rm K} = h/e^2, \tag{10}$$

where h has units J s so that h/e^2 has units (J s)/(A s)^2 = ohm. $R_{\rm K}$ does not involve frequency or angular units which means that if angles are chosen to be dimensioned $R_{\rm K}$ is given by (see also Bunker and Jensen [2])

$$R_{\rm K} = h_{\rm A}/e^2; \tag{11}$$

this involves the Planck constant of action given in Eq. (4) which has the same numerical value as h but with units J s rather than J s cycle^{-1}.

We now derive the expression for the mass M in terms of the Josephson and von Klitzing constants. The voltage V is measured by comparing it to that of a system using the Josephson effect which gives

$$V = n_1 f_1/K_{\rm J}, \tag{12}$$

where n_1 is the number of Josephson junctions in that system and f_1 is a known microwave frequency. To measure I, the current is passed through a resistor and the voltage drop U across the resistance R is measured. I is obtained as the ratio U/R, where U is measured using the same system for a second time, and R is calibrated against a quantum Hall resistor using a resistance bridge. This gives

$$R = rR_{\rm K}, \tag{13}$$

where r is a ratio determined by the resistance bridge. Combining the measurements of U and R we obtain

$$\begin{aligned} I &= U/R \\ &= [n_2 f_2/K_{\rm J}]/[rR_{\rm K}] \\ &= (n_2/r)f_2/[K_{\rm J}R_{\rm K}], \end{aligned} \tag{14}$$

where n_2 and f_2 are from the measurement of the voltage drop U. Substituting Eq. (12) for V and Eq. (14) for I into Eq. (7) for M gives

$$M = [(n_1 n_2)/r][1/(K_{\rm J}^2 R_{\rm K})]f_1 f_2/(gv). \tag{15}$$

If we choose angles to be dimensionless

$$1/(K_{\rm J}^2 R_{\rm K}) = h/4 \tag{16}$$

with units J s, and we deduce

$$M = (n_1 n_2/r)hf_1 f_2/(4\,gv), \tag{17}$$

where h is said to be in units J s and we have to use the values of f_1 and f_2 in cycle s^{-1} although they are said to be in the unit s^{-1} when angles are dimensionless (as in Eq. (6)).

If we choose angles to be dimensioned, the von Klitzing constant is given by Eq. (11), which involves the Planck constant of action $h_{\rm A}$. Because I is the ratio of a voltage measured using the Josephson effect (which involves the Planck constant h in J s cycle^{-1} here) and a resistance measured using the quantum Hall effect (which involves the Planck constant of action $h_{\rm A}$), we obtain an expression involving the ratio $h/h_{\rm A}$. This ratio has the value

$$h/h_{\rm A} = 1.0 \text{ cycle}^{-1}. \tag{18}$$

We now have

$$1/(K_{\rm J}^2 R_{\rm K}) = h(h/h_{\rm A})/4, \tag{19}$$

with units J s cycle^{-2}, to give the following expression for the mass:

$$M = (n_1 n_2/r)h(h/h_A)f_1 f_2/(4\ gv). \tag{20}$$

This equation is our new result. Angles are chosen to be dimensioned; h has units J s cycle^{-1}, h_A has units J s, and frequencies explicitly have units cycle s^{-1} to give M in kg.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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