Levi-Civita symbol and cross product vector/tensor

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1 Definitions

The Levi-Civita symbol ϵ_{ijk} is a tensor of rank three and is defined by

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any two labels are the same} \\ 1, & \text{if } i,j,k \text{ is an even permutation of 1,2,3} \\ -1, & \text{if } i,j,k \text{ is an odd permutation of 1,2,3} \end{cases} \tag{1}$$

The Levi-Civita symbol ϵ_{ijk} is anti-symmetric on each pair of indexes.

The determinant of a matrix A with elements a_{ij} can be written in term of ϵ_{ijk} as

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk} a_{1i} a_{2j} a_{3k} = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \tag{2}$$

Note the compact notation where the summation over the spatial directions is dropped. It is this one that is in use.

Note that the Levi-Civita symbol can therefore be expressed as the determinant, or mixed triple product, of any of the unit vectors $(\hat{\boldsymbol{e}}_1, \hat{\boldsymbol{e}}_2, \hat{\boldsymbol{e}}_3)$ of a normalised and direct orthogonal frame of reference.

$$\epsilon_{ijk} = \det(\hat{\boldsymbol{e}}_i, \hat{\boldsymbol{e}}_j, \hat{\boldsymbol{e}}_k) = \hat{\boldsymbol{e}}_i \cdot (\hat{\boldsymbol{e}}_j \times \hat{\boldsymbol{e}}_k) \tag{3}$$

Now we can define by analogy to the definition of the determinant an additional type of product, the vector product or simply cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{vmatrix} \hat{\boldsymbol{e}}_1 & \hat{\boldsymbol{e}}_2 & \hat{\boldsymbol{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \epsilon_{ijk} \hat{\boldsymbol{e}}_i a_j b_k \tag{4}$$

or for each coordinate

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k \tag{5}$$

2 Properties

- The Levi-Civita tensor ϵ_{ijk} has $3 \times 3 \times 3 = 27$ components.
- $3 \times (6+1) = 21$ components are equal to 0.
- 3 components are equal to 1.
- 3 components are equal to -1.

3 Identities

The product of two Levi-Civita symbols can be expressed as a function of the Kronecker's symbol δ_{ij}

$$\begin{aligned} \epsilon_{ijk}\epsilon_{lmn} &= +\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} \\ &\quad -\delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} \end{aligned} \tag{6}$$

Setting $i = l$ gives

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{7}$$

proof

$$\begin{aligned} \epsilon_{ijk}\epsilon_{imn} &= \delta_{ii}(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{im}\delta_{jn}\delta_{ki} + \delta_{in}\delta_{ji}\delta_{km} - \delta_{im}\delta_{ji}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{ki} \\ &= 3(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) + \delta_{km}\delta_{jn} + \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} - \delta_{kn}\delta_{jm} \\ &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \end{aligned}$$

Setting $i = l$ and $j = m$ gives

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{8}$$

Setting $i = l$, $j = m$ and $k = n$ gives

$$\epsilon_{ijk}\epsilon_{ijk} = 6 \tag{9}$$

Therefore

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b}) \tag{10}$$

proof

$$\begin{aligned} \boldsymbol{d} &= \boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) \\ d_m &= \epsilon_{mni}a_n(\epsilon_{ijk}b_jc_k) \\ &= \epsilon_{imn}\epsilon_{ijk}a_nb_jc_k \\ &= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})a_nb_jc_k \\ &= b_ma_kc_k - c_ma_jb_j \\ &= [\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})]_m - [\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})]_m \end{aligned}$$

In the same way

$$\begin{aligned}
[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{a})]_i &= \epsilon_{ijk}\partial_j\epsilon_{kmn}\partial_m a_n \\
&= \epsilon_{kij}\epsilon_{kmn}\partial_j\partial_m a_n \\
&= \partial_j\partial_i a_j - \partial_j\partial_j a_i \\
&= \partial_i\partial_j a_j - \partial_j\partial_j a_i \\
&= \left[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}\right]_i
\end{aligned}$$

4 Properties

The cross product is a special vector. If we transform both vectors by a reflection transformation, for example a central symmetry by the origin, i.e. $\boldsymbol{v} \to \boldsymbol{v}' = -\boldsymbol{v}$, the cross product vector is conserved.

proof

$$\begin{aligned}
\boldsymbol{p} &= \boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \\
\boldsymbol{p}' &= \boldsymbol{a}'\times\boldsymbol{b}' \\
&= \begin{pmatrix} (-a_2)(-b_3) - (-a_3)(-b_2) \\ (-a_3)(-b_1) - (-a_1)(-b_3) \\ (-a_1)(-b_2) - (-a_2)(-b_1) \end{pmatrix} \\
&= \boldsymbol{p}
\end{aligned}$$

The cross product does not have the same properties as an ordinary vector. Ordinary vectors are called polar vectors while cross product vector are called axial (pseudo) vectors. In one way the cross product is an artificial vector.

Actually, there does not exist a cross product vector in space with more than 3 dimensions. The fact that the cross product of 3 dimensions vector gives an object which also has 3 dimensions is just pure coincidence.

The cross product in 3 dimensions is actually a tensor of rank 2 with 3 independent coordinates.

proof

$$\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})_{ij} &= a_i b_j - a_j b_i = c_{ij} \\
&= \begin{pmatrix} 0 & a_1b_2 - a_2b_1 & a_1b_3 - a_3b_1 \\ a_2b_1 - a_1b_2 & 0 & a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 & a_3b_2 - a_2b_3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -(a_2b_1 - a_1b_2) & a_1b_3 - a_3b_1 \\ a_2b_1 - a_1b_2 & 0 & -(a_3b_2 - a_2b_3) \\ -(a_1b_3 - a_3b_1) & a_3b_2 - a_2b_3 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}
\end{aligned}$$

The correct or consistent approach of calculating the cross product vector from the tensor $(\boldsymbol{a}\times\boldsymbol{b})_{ij}$ is the so called index contraction

$$(\boldsymbol{a}\times\boldsymbol{b})_i = \frac{1}{2}(a_j b_k - a_k b_j)\epsilon_{ijk} = \frac{1}{2}(\boldsymbol{a}\times\boldsymbol{b})_{jk}\,\epsilon_{ijk} \tag{11}$$

proof

$$\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})_i &= \frac{1}{2}c_{jk}\epsilon_{ijk} = c_i \\
&= \frac{1}{2}a_j b_k \epsilon_{ijk} - \frac{1}{2}b_j a_k \epsilon_{ijk} \\
&= \frac{1}{2}(\boldsymbol{a}\times\boldsymbol{b})_i - \frac{1}{2}(\boldsymbol{b}\times\boldsymbol{a})_i \\
&= (\boldsymbol{a}\times\boldsymbol{b})_i
\end{aligned}$$

In 4 dimensions, the cross product tensor is thus written

$$a_i \times b_j = (a_i b_j - a_j b_i) = \begin{pmatrix} 0 & -c_{21} & -c_{31} & -c_{41} \\ c_{21} & 0 & -c_{32} & -c_{42} \\ c_{31} & c_{32} & 0 & -c_{43} \\ c_{41} & c_{42} & c_{43} & 0 \end{pmatrix} \tag{12}$$

This tensor has 6 independent components. There should be 4 components for a 4 dimensions vector, therefore it cannot be represented as a vector.

More generally, if n is the dimension of the vector, the cross product tensor $a_i \times b_j$ is a tensor of rank 2 with $\frac{1}{2}n(n-1)$ independent components.

The cross product is connected to rotations and has a structure which also looks like rotations, called a simplectic structure.