

# Planetary Magnetospheres: Homework Problems

Solutions will be posted online at <http://www.ucl.ac.uk/~ucapnac>

1. In classical electromagnetic theory, the *magnetic moment*  $\mu_L$  associated with a circular current ‘loop’ of radius  $R$  which carries a current  $I$  is given by the product of current and loop area:

$$\mu_L = I \pi R^2.$$

Apply this definition to the current carried by a particle of charge  $q$  and mass  $m$  gyrating in a single plane about a magnetic field of strength  $B$ . The particle thus moves on a circular orbit with speed  $v_\perp$  and radius  $r_g = m v_\perp / (qB)$ . Show that the magnetic moment associated with the current represented by the particle’s motion is equal to the first adiabatic invariant discussed in lectures, i.e.  $\mu = W_\perp / B$ , the ratio of gyrotational kinetic energy to field strength.

2. For an ideal collisionless plasma of bulk velocity  $\mathbf{u}$ , Ohm’s Law reduces to

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B},$$

where  $\mathbf{E}$  is the *convective* electric field. Show that the velocity component perpendicular to  $\mathbf{B}$  is given by  $\mathbf{u}_\perp = \mathbf{E} \times \mathbf{B} / B^2$ .

3. Following on from Question 2, a general plasma flow  $\mathbf{u}$  is sometimes described by its corresponding pattern of convective electric field  $\mathbf{E}$ . If  $\mathbf{E}$  can be described as the gradient of a scalar potential through  $\mathbf{E} = -\nabla\phi_E$ , then we have  $\mathbf{u}_\perp = -\nabla\phi_E \times \mathbf{B} / B^2$ .

Assume for simplicity that  $\mathbf{u}_\parallel = 0$ .

Consider plasma motion in a ‘magnetospheric equatorial’ plane which contains the Earth-Sun line and is perpendicular to the Earth’s magnetic dipole axis. Explain why the ‘streamlines’ of the plasma flow in this plane (curves which have a local tangent vector parallel to  $\mathbf{u}$ ) are also curves of constant  $\phi_E$  (i.e. *equipotential* curves).

In this equatorial plane, we may write  $\phi_E$  as the sum of two terms:

$$\phi_E = \phi_{CR} + \phi_{CONV}.$$

The first term is the corotation potential and dominates close to the planet. It is given by:

$$\phi_{CR} = -\Omega_E B_E R_E^3 / r,$$

where  $\Omega_E$  is the Earth’s angular velocity of rotation,  $B_E$  is the equatorial field strength at the Earth’s surface,  $R_E$  is the Earth’s radius and  $r$  is radial distance from the planet’s centre.

The second term is the convection potential and describes sunward flows (associated with magnetotail reconnection) which carry plasma from the magnetotail towards the dayside:

$$\phi_{CONV} = -E_o y,$$

where  $E_o$  is the convection electric field (assumed constant) and  $y$  is the Cartesian coordinate measured along an axis (lying in the equatorial plane) which passes through the Earth’s centre (the origin) and is perpendicular to the upstream solar wind direction (solar wind flows along the negative  $x$  direction).  $y$  is positive towards dusk.

There is a ‘stagnation’ point in the flow, lying on the positive  $y$  axis, whose location may be estimated as the point where the magnitudes of the two potential terms are equal. Show that the radial distance of the stagnation point is given (in units of Earth radii) by:

$$r_{sp} / R_E = (\Omega_E B_E R_E / E_o)^{1/2}$$

Using reasonable values for the Earth parameters, and a value  $E_O = 1 \text{ mV/m}$ , calculate  $r_{sp}/R_E$  for the Earth's magnetosphere. How does variability in  $E_O$  affect this distance ?

For Jupiter, the planet's very strong field, size and rotation rate cause  $r_{sp}$  to lie *outside* the actual magnetosphere - what is the physical meaning of this result?

4. The magnetic field strength  $B$  due to the Earth's dipole field may be expressed as:

$$B = (B_E R_E^3 / r^3) (3 \cos^2 \theta + 1)^{1/2}, \quad (1)$$

where  $B_E$  is the equatorial field strength at the Earth's surface,  $R_E$  is the Earth's radius and  $r$  is radial distance from the planet's centre.  $\theta$  denotes magnetic colatitude (the magnetic equator is defined by  $\theta = \pi/2$ ).

The following formula is for the pitch angle  $\alpha_c$  associated with the *loss cone* at a point P where the field strength is  $B_P$ :

$$\sin^2 \alpha_c = B_P / B_S, \quad (2)$$

where  $B_S$  is the magnetic field at the surface of the planet which is magnetically connected to the point P along the same field line.

Calculate the value  $\alpha_c$  as a function of distance for locations in the magnetic equatorial plane, using the dipole approximation. You may find the following formula for the shape of a dipole magnetic field line useful:

$$r = LR_E \sin^2 \theta, \quad (3)$$

where  $LR_E$  is the equatorial crossing distance of the field line.

5. The magnetic signatures of interchange observed by *Galileo* in Jupiter's magnetosphere indicate that the inward-moving flux tubes have magnetic field strengths typically higher than the surrounding plasma. If the total (plasma plus magnetic) pressure inside the flux tube is equal to that of the ambient plasma outside, show that the small change in field strength  $\delta B$  (inside minus outside field) is related to a corresponding change in plasma pressure  $\delta p$  as follows:

$$\delta p / p_o = -2(\delta B / B_o)(1 / \beta_o) \quad (4)$$

where the subscript 'o' indicates quantities outside the flux tube, and  $\beta$ , as usual, equals the ratio of plasma pressure to magnetic pressure.

Using this formula, calculate  $\delta p / p_o$  for values: (i)  $B_o = 1700 \text{ nT}$ ,  $\delta B = 10 \text{ nT}$ ,  $\beta_o = 0.05$ ; and (ii)  $B_o = 1700 \text{ nT}$ ,  $\delta B = 25 \text{ nT}$ ,  $\beta_o = 0.05$ .

6. Consider the typical information for Mercury and the Earth in the table from the lecture notes which compares the magnetopause stand-off distances of various planets. Assuming that the dipole magnetic pressure of the planet balances solar wind dynamic pressure at the magnetopause standoff point, calculate the ratio of solar wind dynamic pressures just upstream of Mercury's and the Earth's magnetospheres.
7. Chapman and Ferraro (1930) developed a model of a plasma cloud interacting with the Earth's dipole magnetic field. This model may be applied to investigate the behaviour of the magnetic field generated by the magnetopause currents. In this picture, the Earth's magnetic dipole is situated at the origin (Earth centre) and the dipole axis is orthogonal to the upstream solar wind direction. The magnetopause is then modelled as an infinite conducting plane, perpendicular to the upstream solar wind velocity, and situated a perpendicular distance of  $R_{MP}$  from the planet's dipole axis. Magnetopause currents flow on this plane and generate an *additional* field within the Earth's magnetosphere which is *equivalent* to that of an identical magnetic dipole, known as the '*image dipole*', situated outside the magnetosphere at a distance  $2R_{MP}$  from the Earth's centre along the direction anti-parallel to the upstream solar wind velocity. We define the  $x$  axis to pass through the Earth's centre (where  $x=0$ ) along this direction.

Using this model, calculate and make a plot of the ratio  $B_{TOT}/B_{DIP}$  as a function of distance along the  $x$  axis, from the Earth's surface to the magnetopause plane. Here,  $B_{TOT}$  is the total magnetic field strength due to the actual and image dipoles combined, and  $B_{DIP}$  is the field strength due to the planetary dipole alone.