Planetary Magnetospheres: Solved Problems and Problem Set 1

Solved Problems

1. In classical electromagnetic theory, the *magnetic moment* μ_L associated with a circular current 'loop' of radius R which carries a current I is given by the product of current and loop area:

$$\mu_L = I\,\pi R^2.$$

Apply this definition to the current carried by a particle of charge q and mass m gyrating in a single plane about a magnetic field of strength B. The particle thus moves on a circular orbit with speed $v_\perp$ and radius $r_g = m\,v_\perp/(qB)$. Show that the magnetic moment associated with the current represented by the particle's motion is equal to the first adiabatic invariant discussed in lectures, i.e. $\mu = W_\perp/B$, the ratio of gyrational kinetic energy to field strength.

Solution

Current is charge per unit time which passes a fixed point. For the particle, this may be written $I = q/T$, where T is the gyroperiod, i.e. $I = q^2B/(2\pi m)$. The area of the orbital circle is $A = \pi r_g^2 = \pi m^2 v_\perp^2/(q^2B^2)$. Hence $IA = \frac{1}{2}mv_\perp^2/B = W_\perp/B$.

2. For an ideal collisionless plasma of bulk velocity $\boldsymbol{u}$, Ohm's Law reduces to

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B},$$

where $\boldsymbol{E}$ is the *convective* electric field. Show that the velocity component perpendicular to $\boldsymbol{B}$ is given by $\boldsymbol{u}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2$.

Solution

Using the given definition of $\boldsymbol{E}$, we may write $\boldsymbol{E} \times \boldsymbol{B}/B^2 = (-\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}/B^2$.

Now, $(-\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}/B^2 = (B^2\boldsymbol{u} - (\boldsymbol{B}\cdot\boldsymbol{u})\boldsymbol{B})/B^2$.

If we define a unit vector $\boldsymbol{b} = \boldsymbol{B}/B$, we have $\boldsymbol{E} \times \boldsymbol{B}/B^2 = \boldsymbol{u} - (\boldsymbol{b}\cdot\boldsymbol{u})\,\boldsymbol{b} = \boldsymbol{u} - \boldsymbol{u}_{||} = \boldsymbol{u}_\perp$.

3. Following on from Question 2, a general plasma flow $\boldsymbol{u}$ is sometimes described by its corresponding pattern of convective electric field $\boldsymbol{E}$. If $\boldsymbol{E}$ can be described as the gradient of a scalar potential through $\boldsymbol{E} = -\nabla\phi_E$, then we have $\boldsymbol{u}_\perp = -\nabla\phi_E \times \boldsymbol{B}/B^2$.

Assume for simplicity that $\boldsymbol{u}_{||} = 0$.

Consider plasma motion in a 'magnetospheric equatorial' plane which contains the Earth-Sun line and is perpendicular to the Earth's magnetic dipole axis. Explain why the 'streamlines' of the plasma flow in this plane (curves which have a local tangent vector parallel to $\boldsymbol{u}$) are also curves of constant ϕ_E (i.e. *equipotential* curves).

In this equatorial plane, we may write ϕ_E as the sum of two terms:

$$\phi_E = \phi_{CR} + \phi_{CONV}.$$

The first term is the corotation potential and dominates close to the planet. It is given by:

$$\phi_{CR} = -\Omega_E\,B_E\,R_E^3/r,$$

where Ω_E is the Earth's angular velocity of rotation, B_E is the equatorial field strength at the Earth's surface, R_E is the Earth's radius and r is radial distance from the planet's centre.

The second term is the convection potential and describes sunward flows (associated with magnetotail reconnection) which carry plasma from the magnetotail towards the dayside:

$$\phi_{CONV} = -E_o\,y,$$

where E_o is the convection electric field (assumed constant) and y is the Cartesian coordinate measured along an axis (lying in the equatorial plane) which passes through the Earth's centre (the origin) and is perpendicular to the upstream solar wind direction (solar wind flows along the negative x direction). y is positive towards dusk.

There is a 'stagnation' point in the flow, lying on the positive y axis, whose location may be estimated as the point where the magnitudes of the two potential terms are equal. Show that the radial distance of the stagnation point is given (in units of Earth radii) by:

$$r_{sp}/R_E = (\Omega_E\, B_E\, R_E/E_O)^{1/2}$$

Using reasonable values for the Earth parameters, and a value $E_O = 1\,\mathrm{mV/m}$, calculate r_{sp}/R_E for the Earth's magnetosphere. How does variability in E_O affect this distance ?

For Jupiter, the planet's very strong field, size and rotation rate cause r_{sp} to lie *outside* the actual magnetosphere - what is the physical meaning of this result?

<u>Solution</u> Setting the magnitudes of ϕ_{CONV} and ϕ_{CR} to be equal, and using the fact that the radial distance r is equal to y for a point on the positive y axis, we obtain:

$$\Omega_E\, B_E\, R_E^3/r = E_O r \to (r/R_E) = \sqrt{\Omega_E\, B_E\, R_E/E_O}$$

Using the E_O value given (and transforming to MKS units), a rotation period of 24 hours for the Earth, a radius of 6370 km for the Earth, and $B_E = 3 \times 10^{-5}$ T, we obtain:

$$(r_{sp}/R_E) = \sqrt{\Omega_E\, B_E\, R_E/E_O}$$
$$= \sqrt{(2\pi/(24 \times 3600)) \times 3 \times 10^{-5} \times 6730 \times 10^3/10^{-3}} = 3.83$$

An increase in E_O represents a stronger flow associated with the Dungey cycle, and a consequently smaller stagnation distance, which approximately represents the transition distance from sunward flow in the outer magnetosphere to corotational flow in the plasmasphere.

Jupiter's stagnation point lying outside its magnetosphere means that the dayside equatorial magnetosphere of Jupiter is dominated by rotational flows (more correctly, (sub)corotational with respect to the planet - see the lecture notes).

4. The magnetic field strength B due to the Earth's dipole field may be expressed as:

$$B = (B_E R_E^3/r^3)\,(3\cos^2\theta + 1)^{1/2}, \tag{1}$$

where B_E is the equatorial field strength at the Earth's surface, R_E is the Earth's radius and r is radial distance from the planet's centre. θ denotes magnetic colatitude (the magnetic equator is defined by $\theta = \pi/2$).

The following formula is for the pitch angle α_c associated with the *loss cone* at a point P where the field strength is B_P:

$$\sin^2\alpha_c = B_P/B_S, \tag{2}$$

where B_S is the magnetic field at the surface of the planet which is magnetically connected to the point P along the same field line.

Calculate the value α_c as a function of distance for locations in the magnetic equatorial plane, using the dipole approximation. You may find the following formula for the shape of a dipole magnetic field line useful:

$$r = LR_E\, \sin^2\theta, \tag{3}$$

where LR_E is the equatorial crossing distance of the field line.

Solution For any magnetic equatorial point at distance LR_E, a dipole field line passing through that point will intersect the Earth's surface at a colatitude θ_i given by:

$$\begin{aligned} R_E &= LR_E \sin^2\theta_i \\ \rightarrow \sin\theta_i &= \sqrt{1/L} \\ \rightarrow \cos\theta_i &= \pm\sqrt{(L-1)/L} \end{aligned} \tag{4}$$

Hence the magnetic field magnitude B_S is given by:

$$B_S = (B_E R_E^3/R_E^3)\,(3\cos^2\theta_i + 1)^{1/2} = B_E\,(3(1-1/L)+1)^{1/2}. \tag{5}$$

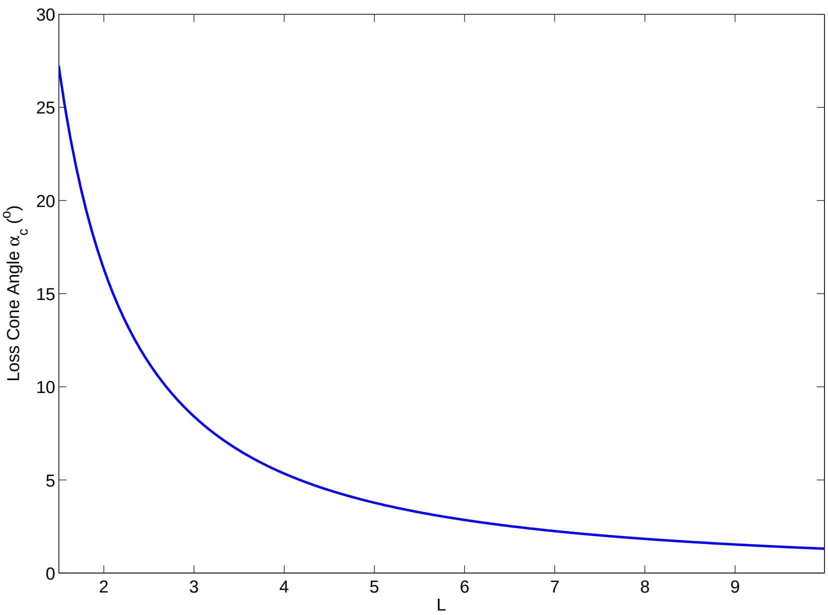
We can also evaluate the dipole formula at $\theta = \pi/2$, $r = LR_E$ to obtain B_P:

$$B_P = B_E/L^3. \tag{6}$$

It follows that:

$$\sin^2\alpha_c = B_P/B_S = L^{-3}\,(3(1-1/L)+1)^{-1/2} \tag{7}$$

Using this formula to evaluate $\sin^2\alpha_c$, hence α_c, as a function of L, we obtain the following plot:



5. The magnetic signatures of interchange observed by *Galileo* in Jupiter's magnetosphere indicate that the inward-moving flux tubes have magnetic field strengths typically higher than the surrounding plasma. If the total (plasma plus magnetic) pressure inside the flux tube is equal to that of the ambient plasma outside, show that the small change in field strength δB (inside minus outside field) is related to a corresponding change in plasma pressure δp as follows:

$$\delta p/p_o = -2(\delta B/B_o)(1/\beta_o) \tag{8}$$

where the subscript 'o' indicates quantities outside the flux tube, and β, as usual, equals the ratio of plasma pressure to magnetic pressure.

Using this formula, calculate $\delta p/p_o$ for values: (i) $B_o = 1700\,\mathrm{nT}$, $\delta B = 10\,\mathrm{nT}$, $\beta_o = 0.05$; and (ii) $B_o = 1700\,\mathrm{nT}$, $\delta B = 25\,\mathrm{nT}$, $\beta_o = 0.05$.

Solution The sum of the magnetic and plasma pressures outside the flux tube may be written as $B_o^2/(2\mu_o) + p_o$. If this quantity remains constant as we cross into the flux tube, we may express this by taking a zero differential between inside and outside, as follows: $d(B^2/(2\mu_o) + p) = 0 \approx 2B_o\,\delta B/(2\mu_o) + \delta p$.

Rearranging and dividing by p_o, we obtain $\delta p/p_o \approx -B_o\,(\delta B/\mu_o)(1/p_o) = -2(\delta B/B_o)(1/\beta_o)$, since, by definition $p_o = \beta_o(B_o^2/(2\mu_o))$.

Using this approximation and the values given, we obtain values of $\delta p/p_o$ of about (i) -0.24 and (ii) -0.59.

6. Consider the typical information for Mercury and the Earth in the table from the lecture notes which compares the magnetopause stand-off distances of various planets. Assuming that the dipole magnetic pressure of the planet balances solar wind dynamic pressure at the magnetopause standoff point, calculate the ratio of solar wind dynamic pressures just upstream of Mercury's and the Earth's magnetospheres.

   Solution

   The table in question indicates that the dipole magnetic pressure at Mercury's dayside magnetopause is approximately proportional to (ignoring dipole tilt effects) $[M_M/(1.4R_M)^3]^2$ (i.e. the magnetic pressure is proportional to the square of the expected field strength). Here M_M is Mercury's magnetic dipole moment. For the Earth, this quantity will be $[M_E/(10R_E)^3]^2$. Taking the ratio, we obtain $(M_M/M_E)^2\,(10^6/1.4^6)(R_E/R_M)^6$. Using reasonable values of the planetary radii, this evaluates to ~ 6.7. (N.B. I think the value of the magnetic moment of Mercury should be more like $4\times10^{-4}\,M_E$, based on Messenger data - note also the usual variability expected in solar wind parameters).

7. Chapman and Ferraro (1930) developed a model of a plasma cloud interacting with the Earth's dipole magnetic field. This model may be applied to investigate the behaviour of the magnetic field generated by the magnetopause currents. In this picture, the Earth's magnetic dipole is situated at the origin (Earth centre) and the dipole axis is orthogonal to the upstream solar wind direction. The magnetopause is then modelled as an infinite conducting plane, perpendicular to the upstream solar wind velocity, and situated a perpendicular distance of R_{MP} from the planet's dipole axis. Magnetopause currents flow on this plane and generate an *additional* field within the Earth's magnetosphere which is *equivalent* to that of an identical magnetic dipole, known as the *'image dipole'*, situated outside the magnetosphere at a distance $2R_{MP}$ from the Earth's centre along the direction anti-parallel to the upstream solar wind velocity. We define the x axis to pass through the Earth's centre (where x=0) along this direction.

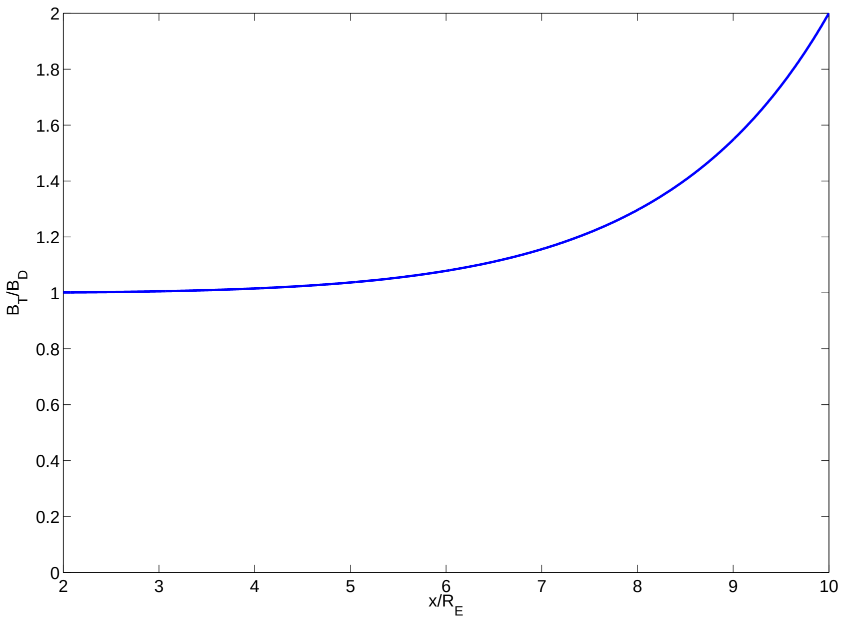
   Using this model, calculate and make a plot of the ratio B_{TOT}/B_{DIP} as a function of distance along the x axis, from the Earth's surface to the magnetopause plane. Here, B_{TOT} is the total magnetic field strength due to the actual and image dipoles combined, and B_{DIP} is the field strength due to the planetary dipole alone.

   Solution

   For the planetary dipole alone, the field strength outside the Earth and inside the magnetopause, along the x axis, is given by the function $B_D(x) = (B_E R_E^3/|x|^3)$ (using the nomenclature of Question 4). Now we may express the field of the image dipole situated at $x = 2R_{MP}$ as the function $B_D(x - 2R_{MP}) = (B_E R_E^3/|x - 2R_{MP}|^3)$. Adding the two, we obtain:

$$B_T(x) = B_D(x)\,(1 + |x|^3/|x - 2R_{MP}|^3).$$

   Hence $B_T(x)/B_D(x) = (1 + |x|^3/|x - 2R_{MP}|^3)$, which is always greater than unity. A plot of this quantity versus x/R_E is given below, using a reasonable value $R_{MP} = 10\,R_E$.



2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
B_T/B_D
2
3
4
5
6
7
8
9
10
x/R_E

Problem Set 1: 'Planetary Magnetospheres' Section

1. Consider the induction equation for an ideal, collisionless plasma threaded by magnetic field $\boldsymbol{B}$, and having bulk flow velocity $\boldsymbol{u}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$$

Consider a continuous 'patch' of plasma (see Notes) which is defined by a surface S, bounded in space by a curve Γ. As the plasma moves, Γ will generally change shape and the area of S will generally change value. Consider an infinitesimally small element of the moving curve Γ which is defined by a vector increment $\boldsymbol{dl}$. Show that, during an infinitesimal time step dt, the motion of this element changes the magnetic flux Φ_B through the patch by an amount:

$$d\Phi_B = \boldsymbol{B} \cdot ((\boldsymbol{u}\, dt) \times \boldsymbol{dl}),$$

where $\boldsymbol{B}$ and $\boldsymbol{u}$ are the local values of field and velocity.

Hence, show that the convective, or *co-moving* time derivative of the magnetic flux through the patch may be written:

$$\frac{D\Phi_B}{Dt} = \frac{\partial}{\partial t}(\Phi_B) + \oint_\Gamma \boldsymbol{B} \cdot (\boldsymbol{u} \times \boldsymbol{dl}),$$

where Φ_B is equal to the surface integral $\iint_S \boldsymbol{B} \cdot \boldsymbol{dS}$.

Making use of an appropriate Maxwell's equation and Ohm's law for the plasma, demonstrate the validity of the 'frozen-in' condition, i.e.

$$\frac{D\Phi_B}{Dt} = 0.$$

2. In Solved Problem 4, you will find the formula for the magnetic field strength the Earth's dipole field, and the equation describing the shape of a dipolar magnetic field line.

The corresponding radial and meridional dipole field components are given by:

$$B_r = -2B_E \cos\theta/(r/R_E)^3$$
$$B_\theta = -B_E \sin\theta/(r/R_E)^3$$

Using this information and appropriate physical constants, calculate the gradient drift velocity $\boldsymbol{u_g} = \frac{W_\perp}{qB^3} \boldsymbol{B} \times \nabla B$ (see Notes) of protons with the following properties, drifting in the Earth's magnetosphere:

(a) $W_\perp = 1\text{keV}$ and 10keV, $r = 8\mathrm{R_E}$, $\theta = 90°$ (i.e. equatorial).

(b) $W_\perp = 1\text{keV}$ and 10keV, $\theta = 60°$, choose r so that proton is on same field line as those in item 1 above.

(c) $W_\perp = 1\text{keV}$ and 10keV, $\theta = 30°$, choose r so that proton is on same field line as those in items 1 and 2 above.

3. In Solved Problem 7, Chapman and Ferraro's 'infinite conducting plane' carries currents which amplify the magnetic field near the Earth's magnetopause by a factor of two. Use the 'pressure balance' argument from the lectures to calculate the change introduced in the standoff distance R_{MP} of a fictitious planet's magnetopause, at fixed solar wind dynamic pressure, when the field is amplified in this way (assume twice the strength of a pure dipole field at the magnetopause).

Consider now adding an interior plasma pressure near the magnetopause of this fictitious planet, such that the plasma β parameter there attains a value 5. What effect does this have on R_{MP} ?

4. Consider a fictitious magnetosphere where rotational effects are not important, and the only forces are due to the plasma pressure gradient and the magnetic $\boldsymbol{J} \times \boldsymbol{B}$ force. If the system is in perfect force balance (i.e. the sum of these two forces at any point is identically zero), explain why the plasma pressure will be uniform all the way along a magnetic field line ?

   Now consider an idealized magnetosphere where rotation plays an important role in force balance, and the magnetic field is symmetric about the rotational / magnetic equatorial plane. The magnetic force, centrifugal force and plasma pressure gradient always add to zero at any point in the system (we neglect all other forces for simplicity). By considering force balance in the direction parallel to the poloidal magnetic field (zero B_ϕ), explain why the addition of the centrifugal force on the plasma causes plasma pressure to *change* along the magnetic field line. Demonstrate why the profile of the plasma pressure can be described by the equation:

$$P(\rho) = P_0 \exp[(\rho^2 - \rho_0^2)/(2l^2)], \tag{9}$$

   where $\rho = r \sin\theta$ denotes cylindrical radial distance, the field line crosses the equator at $\rho = \rho_0$, and the scale length $l \approx (2kT/m_i\omega^2)^{1/2}$. Assumptions: the plasma temperature T and angular velocity ω are constant along a field line; the plasma is quasi-neutral, behaves as an ideal gas, and is composed of ions of mass m_i and electrons of mass m_e.

Solutions

1. The element $\boldsymbol{dl}$ changes position by $\boldsymbol{u}dt$ in the time step. The corresponding surface area covered by the element during this motion is thus a parallelogram having these vectors as edges, and may thus be written $\boldsymbol{dS} = (\boldsymbol{u}\,dt) \times \boldsymbol{dl}$ - here the usual convention is followed, where a surface element is represented by a vector lying orthogonal to itself. $d\Phi_B$, by definition, is the scalar product of magnetic field and surface vector, i.e. the flux of magnetic field through the surface.

   The $\oint$ integral represents the change in Φ_B due to the motion of all of the elements $\boldsymbol{dl}$ which make up the moving perimeter Γ.

   In general, however, the magnetic field itself will have an explicit time dependence - i.e. an observer at a fixed point in space will see $\boldsymbol{B}$ change with time. Due to this effect, the change in Φ_B can be written

$$d\Phi_B' = dt \int\int_S \frac{\partial \boldsymbol{B}}{\partial t}\,\boldsymbol{dS}$$

   .

   The co-moving derivative is:

$$\frac{d\Phi_B}{dt} + \frac{d\Phi_B'}{dt},$$

   which is:

$$\begin{aligned} & \oint_\Gamma \boldsymbol{B}\cdot(\boldsymbol{u}\times\boldsymbol{dl}) + \int\int_S \frac{\partial \boldsymbol{B}}{\partial t}\,\boldsymbol{dS} \\ = & \oint_\Gamma \boldsymbol{dl}\cdot(\boldsymbol{B}\times\boldsymbol{u}) + \int\int_S -\nabla\times\boldsymbol{E}\cdot\boldsymbol{dS} \\ = & \oint_\Gamma \boldsymbol{dl}\cdot(\boldsymbol{B}\times\boldsymbol{u}) + \oint_\Gamma -\boldsymbol{E}\cdot\boldsymbol{dl} \end{aligned}$$

   where $\boldsymbol{E}$ denotes the electric field, and we have used

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

   Using the idealized Ohm's Law $\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B}$, we obtain:

$$\frac{d\Phi_B}{dt} = \oint_\Gamma (\boldsymbol{B}\times\boldsymbol{u})\cdot\boldsymbol{dl} + \oint_\Gamma (\boldsymbol{u}\times\boldsymbol{B})\cdot\boldsymbol{dl},$$

   which is zero.

2. If I haven't made any errors, the evaluation of $\boldsymbol{B}\times\nabla B$ gives (help from Mathematica!):

$$\frac{-3B_E^2 R_E^6 \sin\theta(1+\cos^2\theta)}{r^7\sqrt{(1+3\cos^2\theta)}}\,\hat{\boldsymbol{\phi}}$$

   To obtain $\boldsymbol{u_g}$, we multiply this expression by $W_\perp/(qB^3)$ and obtain:

$$\begin{aligned} (W_\perp/q)\frac{-3r^2\sin\theta(1+\cos^2\theta)}{B_E R_E^3(3\cos^2\theta+1)^2}\,\hat{\boldsymbol{\phi}} = \\ (W_\perp/q)\frac{-3(LR_E\sin^2\theta)^2\sin\theta(1+\cos^2\theta)}{B_E R_E^3(3\cos^2\theta+1)^2}\,\hat{\boldsymbol{\phi}} \end{aligned} \tag{10}$$

   Here we have eliminated r using the dipole field line formula ($L = 8$ for this problem).

   If we use appropriate values $R_E = 6370\text{km}$, and $B_E = 3\times10^{-5}$ T, we obtain the following values for the energy $W_\perp = 1\text{keV}$: $|\boldsymbol{u_g}| \approx 1005\text{m/s}(\theta = 90°), 1.05\text{m/s}(\theta = 60°), 0.524\text{m/s}(\theta = 30°)$. For the case $W_\perp = 10\text{keV}$, multiply these values by ten.

   (This problem requires much algebra, so please let me know if you spot any mistakes!)

3. Balancing magnetic pressure of a pure dipole with the solar wind dynamic pressure:

$$B_{DIP}(R_{MP})^2/(2\mu_o) = \frac{1}{2\mu_o}\left(\frac{B_E R_E^3}{R_{MP}^3}\right)^2 = P_{SW}$$
$$R_{MP} = \left(\frac{1}{2\mu_o}\right)^{1/6}\left(\frac{B_E^2 R_E^6}{P_{SW}}\right)^{1/6} \tag{11}$$

Looking at this equality, we see that if we replace $B_{DIP}(R_{MP})$ by $2\,B_{DIP}(R_{MP})$, then R_{MP} will increase from the pure dipole value by a factor $4^{1/6} \approx 1.26$.

If we now introduce the plasma β value as well (ratio of plasma pressure to magnetic pressure), then the total pressure (plasma plus magnetic) at the magnetopause can be written:

$$(1+\beta)(2\,B_{DIP}(R_{MP}))^2/(2\mu_o)$$

So the pressure balance becomes:

$$(1+\beta)(2\,B_{DIP}(R_{MP}))^2/(2\mu_o) = \frac{1}{2\mu_o}\,(1+\beta)\left(\frac{2B_E R_E^3}{R_{MP}^3}\right)^2 = P_{SW}$$
$$R_{MP} = \left(\frac{1}{2\mu_o}\right)^{1/6}(1+\beta)^{1/6}\,4^{1/6}\left(\frac{B_E^2 R_E^6}{P_{SW}}\right)^{1/6}$$

Hence the non-zero plasma pressure increases R_{MP} by an additional factor $(1+\beta)^{1/6} = 6^{1/6} \approx 1.35$.

4. The equation of force balance parallel to the magnetic field is:

$$\frac{-dP}{ds} + \frac{N}{2}(m_i + m_e)\rho\omega^2\cos\psi = 0,$$

where N is total particle number density and ψ is the angle between the field direction and the cylindrical radial direction (i.e. the local direction perpendicularly outwards from the axis of symmetry). Note that we don't need to consider any other force, since the parallel component of $\boldsymbol{J} \times \boldsymbol{B}$ is zero, by definition. Since the centrifugal term always points outwards (positive direction), we require $\frac{-dP}{ds}$ to be negative, i.e. pressure P must increase as we travel along a field line from polar regions to equator (confinement of plasma into a disc-like shape).

Since an element of length ds along the field corresponds to a change $d\rho = ds\,cos\psi$, we have:

$$\frac{-dP}{d\rho} + \frac{P}{2kT}(m_i + m_e)\rho\omega^2 = 0,$$

where $P = NkT$ for the plasma.

Integrating between an arbitrary point on the field line and the equator (denoted by subscript '0'):

$$\int \frac{dP}{P} = \int \frac{(m_i + m_e)\omega^2}{2kT}\rho\,d\rho,$$
$$\ln(P_0/P) = \frac{(m_i + m_e)\omega^2}{2kT}\,\frac{1}{2}(\rho_0^2 - \rho^2)$$
$$P = P_0\,\exp[(\rho^2 - \rho_0^2)/(2l^2)],$$

where $l^2 = \frac{2kT}{(m_i+m_e)\omega^2} \approx \frac{2kT}{m_i\omega^2}$, since $m_i >> m_e$.

Problem Set 2

1. Explain why the volume of a unit magnetic flux tube (i.e. the volume per unit magnetic flux) is given by the integral $\int \frac{ds}{B}$ along the field line, where ds is length element along the field, and $B(s)$ is local field strength.

   Consider now a cold plasma (quasi-neutral, with one species of positive ion) in a rotating magnetosphere (as in Problem Set 1). Show that the number of ions N_i contained per unit magnetic flux can be expressed as:

   $$N_i = \frac{P_o}{2kT} \int \exp[(\rho^2 - \rho_o^2)/(2l^2)] \frac{ds}{B}, \tag{12}$$

   where the integral is again along the field line, pressure is denoted by P, cylindrical radial distance by ρ, and quantities at the magnetic equator are subscripted with 'o'. l is the length scale from Problem Set 1, which involves the temperature T and plasma angular velocity ω, both constant along the field line.

2. Derive the first-order density and temperature perturbations given in the section on 'Interchange Motions':

   $$\sigma n^{(1)} = -n \nabla \cdot \mathbf{u} - \mathbf{u} \cdot \nabla n$$
   $$\sigma P^{(1)} = -\gamma P (\nabla \cdot \mathbf{u}) - \mathbf{u} \cdot \nabla P$$

   You may find it useful to consider the perturbed equations for conservation of mass, and for adiabatic change in plasma pressure (this second condition may be expressed as $\frac{D(Pn^{-\gamma})}{Dt} = 0$ - a zero comoving time derivative). For simplicity, assume that the unperturbed plasma has zero velocity.

3. Consider a spherically symmetric inward flow of material being accreted onto a star of mass M. Assume that the material is freely falling under the influence of the star's gravity, starting from rest at infinite distance. The accretion rate $\dot{M}$ is constant and equal to $4\pi r^2 \rho_M v$, where r is radial distance from the star's centre, ρ_M is density of the material and v is the velocity. Explain why this equality is valid in the steady-state flow.

   Assume now a very simplified estimate for the magnetic field strength for the star, based on a dipole's radial dependence: $B(r) \sim \mu/r^3$, where μ is the star's magnetic moment (we ignore the angular dependence for simplicity).

   Using this information, show that the approximate Alfvèn radius R_A of the system, where the dynamic pressure of the inflow ($\rho_M v^2$) equals the magnetic pressure of the star, satisfies the dependence:

   $$R_A \propto \mu^{4/7} \dot{M}^{-2/7} M^{-1/7}$$

Problem Set 3

1. The 'propeller' mechanism may act to eject infalling material from the magnetospheric boundary of a rapidly rotating, magnetized star.

   In a simple picture, material instantaneously 'attaches' to the rotating field at the magnetospheric radius R_μ and starts to rotate with the stellar angular velocity Ω_S. The propeller mechanism is effective when the velocity of the 'attached' material exceeds the local escape velocity from the star. Show that this condition is equivalent to:

   $$R_\mu > 2^{1/3} R_c,$$

   where R_c is the corotation radius (i.e. the radius at which the angular velocity of a circular orbit about the star is equal to Ω_S).

2. Consider a Polar binary star system where the magnetic dipole of the white dwarf is orthogonal to the orbital plane of the two stars. Assume that the magnetic field at and inside the coupling region, where accreting material starts to flow along field lines, is dominated by the dipolar field of the white dwarf.

   Calculate the range in radial distance (in units of white dwarf radii) covered by the coupling region corresponding to an 'arc-shaped' accretion shock on the white dwarf surface extending between magnetic colatitudes of 20° and 28°. By what factor does the field strength change over the coupling region?

   Consider an individual 'blob' in the accreting material which is channelled by the magnetic field onto the white dwarf surface. δA represents the 'cross-section' area of the blob, locally perpendicular to the field. Estimate, using a dipole field model, the factor by which δA changes as the blob falls from the inner edge of the coupling region to the white dwarf surface.

   (If radial distance is denoted by r and magnetic colatitude by θ, the equation of a dipole field line is $r = LR_{wd} \sin^2 \theta$, where LR_{wd} is the distance at which the field line crosses the magnetic equator. The magnetic field strength due to the dipole is proportional to the quantity $r^{-3} \left(1 + 3\cos^2\theta\right)^{1/2}$).

3. Consider a Polar system with a single active accretion shock which emits electron cyclotron radiation. 'Peaks' in the continuum emission of the system occur at wavelengths corresponding to harmonics of the electron cyclotron frequency. If two of the adjacent harmonic peaks occur at optical wavelengths of 7146 Å and 6125 Å, estimate the magnetic field strength at the location of the emission region on the white dwarf surface.

4. Consider Ghosh and Lamb's picture of the magnetic torque acting between a magnetized, accreting star and its surrounding accretion disc. What would happen to the corotation radius following an unusual transient episode of strongly enhanced accretion, which 'spins up' the star to a higher angular velocity? If the magnetospheric radius R_μ instantly returns to its 'quiet' value immediately following this episode, but now the corotation radius lies inside $2^{-1/3} R_\mu$. What would happen to the rate of accretion onto the star's surface? What would happen to the areas of the disc which are attached to 'forward-swept' and 'backswept' field lines? What would be the consequences for the spin rate of the star?