QPTs in the dispersive driven dissipative Jaynes-Cummings model

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Brief Outline of Tools and Models

- Master Equation and Single Quantum Trajectories.
- The Piecewise Deterministic Process and the Fokker-Planck equation.
- The Jaynes-Cummings Model and the $\sqrt{n}$ oscillator:
  - I. At resonance
  - II. Dispersive regime
- The dispersive strongly driven generalized Jaynes-Cummings oscillator.

The Piecewise Deterministic Process I

- The density matrix $\rho_S(t)$ of an open quantum system satisfies the Markovian master equation (in Lindblad form) [1]

$$\frac{d}{dt}\rho_S(t) = -i[H, \rho_S(t)] + \sum_i \gamma_i \left(A_i \rho_S(t) A_i^\dagger - \frac{1}{2} A_i^\dagger A_i \rho_S(t) - \frac{1}{2}\rho_S(t) A_i^\dagger A_i \right) \tag{1}$$

- Liouville master equation for a PDP in Hilbert space:

$$\frac{\partial}{\partial t} P[\psi, t] = i \int dx \left\{ \frac{\delta}{\delta\psi(x)} G(\psi)(x) - \frac{\delta}{\delta\psi^*(x)} G(\psi)^*(x) \right\} P[\psi, t] + \int D\tilde{\psi} D\tilde{\psi}^* \{ W[\psi|\tilde{\psi}] P[\tilde{\psi}, t] - W[\tilde{\psi}|\psi] P[\psi, t] \}, \tag{2}$$

- **A)** Deterministic evolution under the action of the operator:

$$G(\psi) = \hat{H}\psi + \frac{i}{2}\sum_i \gamma_i \|A_i\psi\|^2 \psi. \tag{3}$$

[1] Breuer and Petruccione, The Theory of Open Quantum Systems, Oxford 2006

The Piecewise Deterministic Process II

- The linear operator $\hat{H}$ is defined as:

$$\hat{H} = H - \frac{i}{2} \sum_i \gamma_i A_i^\dagger A_i. \tag{4}$$

- The last term is added in order to preserve the norm of the wavefunction, leading to the ultimate unravelling of the master equation.

- The operator $\hat{H}$ is non-Hermitian. It reflects the decaying norm of the wavefunction corresponding to no photon count.

- **B)** The transition rate (stochastic term) is

$$W[\psi|\tilde{\psi}] = \sum_i \gamma_i ||A_i\tilde{\psi}||^2 \, \delta \left[\frac{A_i\tilde{\psi}}{||A_i\tilde{\psi}||} - \psi \right]. \tag{5}$$

The Piecewise Deterministic Process III

- Assume one jump operator A with a corresponding rate γ_0. A diffusion expansion of a given master equation can be realized if
  - the size of the transitions among the states becomes arbitrarily small and
  - the number of transitions in any finite time interval becomes arbitrarily large.

$$A = I + \varepsilon C, \tag{6}$$

- With these expressions, the Fokker-Planck equation is equivalent to the following stochastic Schroedinger equation in Itô form:

$$d\psi(t) = -iK(\psi(t))dt + \sqrt{\gamma}M(\psi(t))dW(t), \tag{7}$$

- $dW(t)$ is an increment of a real Wiener process.

The driven Jaynes Cummings Model

$$H_{JC} = \frac{1}{2}\hbar\omega_A\sigma_z + \hbar\omega_c a^\dagger a + i\hbar g\left(a^\dagger\sigma_- - a\sigma_+\right) + \hbar\left(\overline{\mathcal{E}}_0 e^{-i\omega_0 t} a^\dagger + \overline{\mathcal{E}}_0^* e^{i\omega_0 t} a\right). \tag{8}$$

- Two competing interactions: the JC interaction between the atom and the cavity mode, and the interaction of the cavity mode with the external driving field. [2]
- In resonance fluorescence the bare atomic levels split as a result of the atom-field interaction:

$$H_{RF} = \frac{1}{2}\hbar\omega_A\sigma_z + \hbar\omega_A a^\dagger a + \hbar\left(\kappa a\sigma_+ + \kappa^* a^\dagger\sigma_-\right) \tag{9}$$

- The new energies of the dressed states are

$$E_{n,\pm} = \left(n + \frac{1}{2}\right)\hbar\omega_A \pm \sqrt{n+1}\,\hbar\,|\kappa|\,. \tag{10}$$

- We anticipate 'dressing' of the 'dressed states'. The threshold for spontaneous dressed-state polarization occurs at $2\left|\overline{\mathcal{E}}_0\right| = g$.

[2] Carmichael, Statistical Methods in Quantum Optics, 2, Springer 2008

Master Equation, Wigner and Q representations

- The Master Equation with dissipation (at rates 2κ for the cavity photons and γ, γ_ϕ for the atom) is:

$$\begin{aligned}\dot{\rho} = &-(i/\hbar)[H_{JC}, \rho] + \kappa[\bar{n}(\omega_c) + 1]\mathcal{L}\{a, \rho\} + \kappa\bar{n}(\omega_c)\mathcal{L}\{a^\dagger, \rho\} + \\ &+ (\gamma/2)[\bar{n}(\omega_q) + 1]\mathcal{L}\{\sigma_-, \rho\} + (\gamma/2)\bar{n}(\omega_q)\mathcal{L}\{\sigma_+, \rho\} + \\ &+ (\gamma_\phi/2)\mathcal{L}\{\sigma_z, \rho\},\end{aligned} \tag{11}$$

  where $\mathcal{L}\{B, \rho\} = 2B\rho B^\dagger - B^\dagger B\rho - \rho B^\dagger B$.

- Characteristic functions

$$\chi_A(z, z^*) = \mathrm{tr}(\rho e^{iza} e^{iz^* a^\dagger}) \quad \text{and} \quad \chi_S(z, z^*) = \mathrm{tr}(\rho e^{iza + iz^* a^\dagger}). \tag{12}$$

- Their Fourier transforms

$$Q, W(\alpha, \alpha^*) = \int \chi_{A,S}(z, z^*) e^{-iz^* \alpha^*} e^{-iz\alpha}\, d^2 z. \tag{13}$$

  are quasi-probability density functions.

The resonant case (Energy spectrum)

At resonance $\omega_0 = \omega_A = \omega_c$:

- For weak driving fields $(2\left|\overline{\mathcal{E}}_0\right|/g \ll 1)$ we expect the Jaynes-Cummings spectrum plus a perturbative correction of order $(2\left|\overline{\mathcal{E}}_0\right|/g)^2$.

- For strong driving fields $(2\left|\overline{\mathcal{E}}_0\right|/g \gg 1)$ the rôles of the interactions are reversed: *the JC interaction becomes the perturbation*.

- The second order phase transition is expected at $2\left|\overline{\mathcal{E}}_0\right| = g$.
- With drive-cavity detuning the large mean photon number is

$$n = \frac{\varepsilon^2}{\kappa^2 + [(\omega_0 - \omega_c) \mp g/(2\sqrt{n})]^2}. \tag{14}$$

The resonant case (Energy spectrum)

- Quasi-energies are associated to a time-independent Scrödinger equation with Hamiltonian:

$$\tilde{H} = i\hbar g \left(a^{\dagger}\sigma_{-} - a\sigma_{+} \right) + \hbar \left(\overline{\mathcal{E}}_0 a^{\dagger} + \overline{\mathcal{E}}_0^{*} a \right) \tag{15}$$

- For strong driving fields ($2\left|\overline{\mathcal{E}}_0\right|/g \gg 1$) the roles of the interactions are reversed: the JC interaction becomes the perturbation.
- Potential energy $\hbar\overline{\mathcal{E}}_0(a^{\dagger} + a)$:proportional to the position of a harmonic oscillator. **We anticipate a continuous spectrum**.

- The transition is anticipated at $2\left|\overline{\mathcal{E}}_0\right| = g$.

The resonant case (Energy spectrum)

- Quasi-energies $E_{n,\pm} = \pm\sqrt{n}\,\hbar g[1 - (2\left|\overline{\mathcal{E}}_0\right|/g)^2]^{3/4}$.
- Spectrum $(m - 1/2)\omega_A + E_{n,\pm}/\hbar$, $m = 0, 1, 2, \ldots$ below threshold.
- Define a squeeze parameter (think of the parametric oscillator):

$$e^{-2r} \equiv \sqrt{1 - \left(2\left|\overline{\mathcal{E}}_0\right|/g\right)^2}. \qquad (16)$$

- For arbitrary detunings the JC Hamiltonian acquires the term $\frac{1}{2}\hbar\Delta\omega_A\sigma_z + \hbar\Delta\omega_c a^\dagger a$.

The resonant case (Two ladders)

- Two quasi-annihilation operators U and L for two ladders beginning from the same ground state.

- The JC Hamiltonian can be written as:

$$H_S + \frac{1}{2}\hbar\omega_A = 0\,|G\rangle\langle G| + \left(\hbar\omega_A U^\dagger U + \hbar g\sqrt{U^\dagger U}\right) + \\ + \left(\hbar\omega_A L^\dagger L - \hbar g\sqrt{L^\dagger L}\right) + \hbar\left(\overline{\mathcal{E}}_0 a^\dagger + \overline{\mathcal{E}}_0^* a\right) \tag{17}$$

- Two $\sqrt{n}$ anharmonic oscillators driven away from resonance.

- For a small cavity damping we form the Master Equation (U)-oscillator

$$\dot{\rho} = \frac{1}{i\hbar}\left[H^{+}_{\sqrt{n}},\, \rho\right] + \kappa\left(2U^\dagger\rho U - U^\dagger U\rho - \rho U^\dagger U\right) \tag{18}$$

The resonant case (Two regimes)

- Weak excitation limit ($\sqrt{n}$ oscillator):

$$\langle a^\dagger a \rangle_{ss} \approx \left| \frac{\overline{\mathcal{E}}_0}{\kappa + ig} \right|^2 \approx \left(\frac{|\overline{\mathcal{E}}_0|}{g} \right)^2 . \tag{19}$$

- Strong excitation-quasi resonant, with detuning $g/(2\sqrt{n})$ (for the $\sqrt{n}$ oscillator):

$$\langle a^\dagger a \rangle_{ss} \approx \left| \frac{\overline{\mathcal{E}}_0}{\kappa + ig/(2\sqrt{\langle a^\dagger a \rangle_{ss}})} \right|^2 \approx \left(\frac{|\overline{\mathcal{E}}_0|}{\kappa} \right)^2 - \left(\frac{g}{2\kappa} \right)^2 . \tag{20}$$

- Mean-field equations for zero system-size (scale of $n_{\text{sat}} = \gamma^2/(8g) \to 0$) predict above threshold (JC):

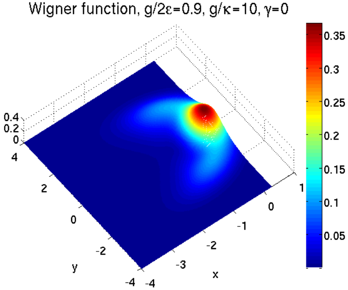
$$A^2 = \left(\frac{|\overline{\mathcal{E}}_0|}{\kappa} \right)^2 \left[1 - \left(\frac{g}{2\,|\overline{\mathcal{E}}_0|} \right)^2 \right] \tag{21}$$

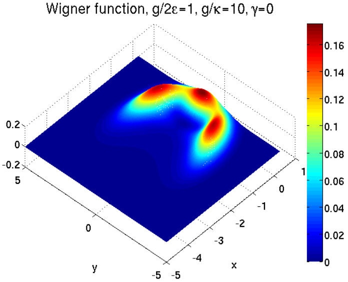
The resonant case (Stability of the Stationary States)

- Zero-system size: for one atom and strong coupling the semiclassical stationary states are "attractors".
- This is corroborated by the Wigner function (two Gaussians centred at $\alpha_{\text{semiclassical}}$).
- Well above threshold, in the presence of spontaneous emission ($\gamma \neq 0$) we find an amalgamation of **(a)** spontaneous dressed state polarization and **(b)** absorptive optical bimodality.
- A spontaneous emission event places the atom in a superposition of $|U\rangle$ and $|L\rangle$ states. The system is not localized on either branch. If the polarization switches, then the phase of the intracavity field will also switch. A "skirt" connecting the peaks appears.

The resonant case (Quantum Fluctuations-I)

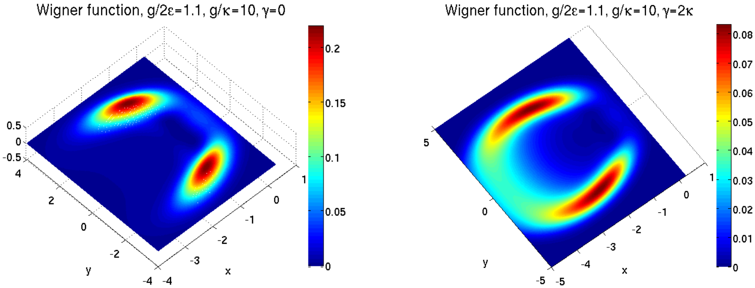
- Quantum fluctuations perform the symmetry breaking.
- The mean photon number behaves similarly to the $\sqrt{n}$ oscillator, but the fluctuations are markedly different.





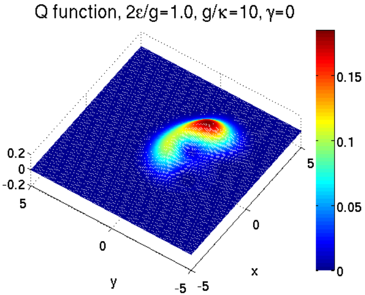
The resonant case (Quantum Fluctuations-II)

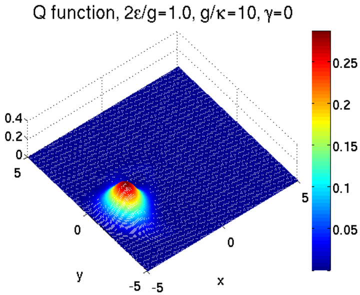
- Above threshold the internal coupling is not sufficiently strong to produce energy splitting and hence levels in excess of the level widths.
- The treatment of semi-classical dynamics plus “small fluctuations” is inapplicable.



The resonant case (Quantum Fluctuations-III)

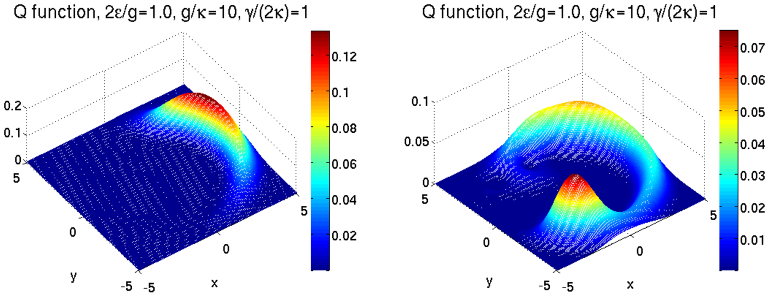
- Single Trajectory for $\gamma = 0$. The state follows the contour specified by the Wigner function, marking the fluctuations.





The resonant case (Quantum Fluctuations-IV)

- Single Trajectory for $\gamma = 2\kappa$. The "skirt" of the Wigner function is unravelled.



The resonant case (Quantum Fluctuations-V)

- The state of the cavity field obeys the stochastic equation:

$$\frac{d\tilde{\alpha}}{dt} = -[\kappa + i\epsilon g/(2|\tilde{\alpha}|)]\tilde{\alpha} + i\overline{\mathcal{E}}_0, \tag{22}$$

- $\epsilon = \pm 1$ representing the random switching events.
- In the strong coupling the JC interaction term gives rise to the operator $d_z = i(\sigma_- - \sigma_+)$. This in turn generates the coupling term

$$\dot{\rho}_{A\alpha} = -ig/(2\sqrt{\overline{n}})\frac{1}{2}\left(d_z[a^\dagger a, \rho] + [a^\dagger a, \rho]d_z\right). \tag{23}$$

- Performing the secular transformation yields the "switching terms"

$$... + \gamma/4\left(d_-\tilde{\rho}d_+ + d_+\tilde{\rho}d_-\right) + ... \tag{24}$$

with $d_+ = |u\rangle\langle l|$ and $d_- = |l\rangle\langle u|$

- These terms couple the U and L paths.

The resonant Case (Forming a Quantum Trajectory)

- Emission times: $t_1, t_2, \ldots t_N$. [3]

- S: **collapse operator** and $e^{(L-S)(t_j - t_{j-1})}$: the **propagator**

$$\tilde{\rho}_c = \begin{cases} \dfrac{e^{(L-S)(t-t_{j-1})}\tilde{\rho}_c(t_{j-1})}{\mathrm{tr}[e^{(L-S)(t-t_{j-1})}\tilde{\rho}_c(t_{j-1})]}, & t_{j-1} \leq t < t_j \\ \dfrac{Se^{(L-S)(t-t_{j-1})}\tilde{\rho}_c(t_{j-1})}{\mathrm{tr}[Se^{(L-S)(t-t_{j-1})}\tilde{\rho}_c(t_{j-1})]}, & t = t_j. \end{cases} \tag{25}$$

- with $S\,O = (\gamma/4)\,(d_- O d_+ + d_+ O d_-)$ and

- $(L - S)O$: determining evolution between switching events (deterministic time evolution).

[3] Alsing and Carmichael, Quantum Opt., 1991

- Following a transformation the decouples the qubit from the cavity, we obtain (keeping the same form for the transformed drive): [4]

$$H = \hbar\omega_c a^\dagger a + \hbar(\omega_c - \Delta)\frac{1}{2}\sigma_z + \hbar\left(\overline{\mathcal{E}}_0 a^\dagger + \overline{\mathcal{E}}_0^* a\right) \tag{26}$$

- $\Delta = \sqrt{\delta^2 + 4g^2 N}$, with $\delta = \omega_A - \omega_c$ the detuning and
- $N = a^\dagger a + \frac{1}{2}\sigma_z + \frac{1}{2}$: total number of excitations.

- 'Decoupled' quantum master equation:

$$\dot{\rho} = -i[H, \rho] + \frac{\kappa}{2}\left([a\rho, a^\dagger] + [a, \rho a^\dagger]\right). \tag{27}$$

- To avoid photon blockade, $N \gg g^4/(\kappa\delta^3)$. The semiclassical model is invalidated by the non convergence of the root expansion in powers of N/N_{crit}, where $N_{\text{crit}} = \delta^2/(4g^2)$.

[4] Bishop, Ginossar and Girvin, PRL, 2010

The dispersive regime II

- Viewed as a saturable extension of the Duffing model [5]:

$$\alpha = -i\mathcal{E}_0 \left\{ \kappa - i \left[(\omega_0 - \omega_c) - \frac{g^2}{\delta} \left(1 + \frac{4g^2}{\delta^2} |\alpha|^2 \right)^{-1/2} \right] \right\}^{-1} \quad (28)$$

- In the dispersive regime $g^2/\delta^2 \ll 1$, hence the nonlinearity manifests in the high photon number regime.

- This abides by the mean-field absorptive optical bistability where the nonlinearity is brought into play for photon numbers:

$$|\alpha|^2 \sim n_{\text{sat}} = \gamma^2/(8g^2). \quad (29)$$

- Markedly different from the resonant case where the magnitude of the Bloch vector is preserved and the nonlinearity diverges for small photon numbers for zero system size.

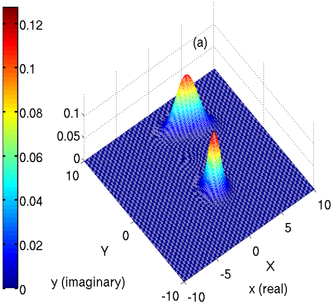
[5] Carmichael, Phys. Rev. X, 2015

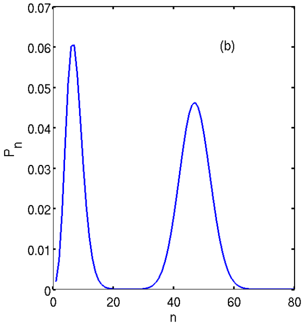
The dispersive regime III (The Generalized JC model)

$$H_{\mathrm{GJC}} = \omega_c a^\dagger a + \sum_n \omega_n |n\rangle\langle n| + \sum_{m,n} g_{mn} |m\rangle\langle n| (a + a^\dagger) + i\varepsilon_d (a^\dagger e^{-i\omega_d t} - a e^{i\omega_d t}). \quad (30)$$

Duffing approximation for a transmon: $\omega_n = \chi n(1-n)$.

Region of cavity bistability:





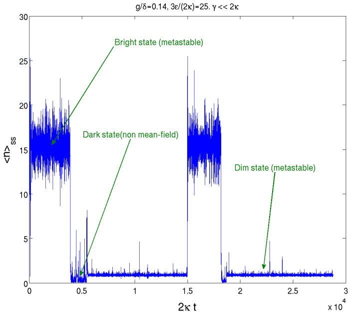
The Q representation for a damped harmonic oscillator

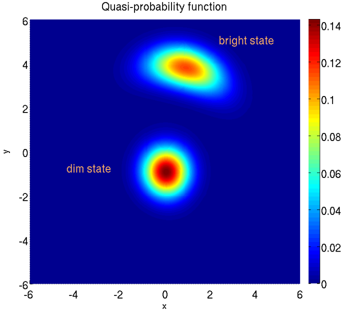
- Time-evolution for a damped coherent state.
- Harmonic oscillator coupled to a thermal bath:

$$Q(\alpha, \alpha^*, t)_{\rho(0)=|\alpha_0\rangle\langle\alpha_0|} = \\ = \frac{1}{\pi[1+\overline{n}(1-e^{-\gamma t})]} \exp\left[-\frac{|\alpha - \alpha_0 e^{-(\gamma/2)t} e^{-i\omega_0 t}|^2}{1+\overline{n}(1-e^{-\gamma t})}\right] \tag{31}$$

- Fluctuations have a quantum mechanical origin. Finite width even when $\overline{n} = 0$.
- During switching there is a 'spiral rotation' of the two states in the phase diagram.

- Bimodality in the photon number $\langle a^\dagger a \rangle$ and the corresponding Q-function.



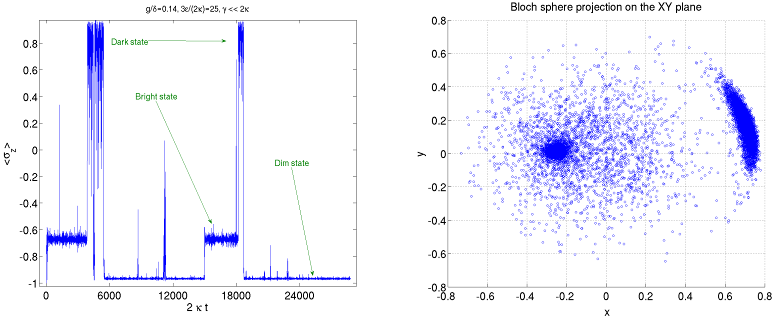


- Switching rates are of the same order, marking a diffuse kinetic first-order transition [6].

[6] Dykman and Smelyanskii, Sov. Phys. JETP, 1988

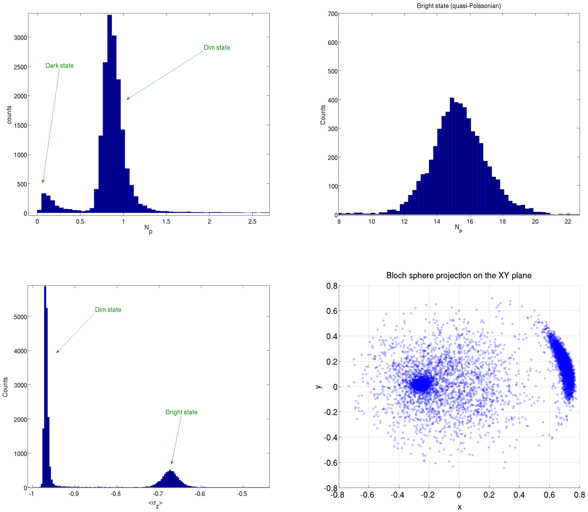
The dispersive regime (Single Quantum Trajectory II)

- Bimodality in the spin inversion $\langle \sigma_z \rangle$ and the corresponding Bloch sphere.



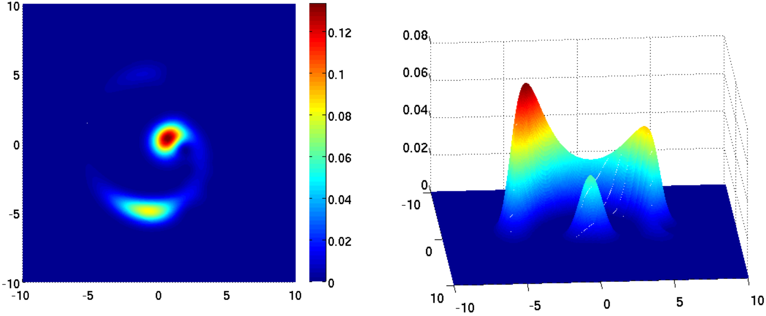
- Compare to the Q-function for the intracavity photons:coherent cancellation.

- Poissonian distribution corresponding to the two coherent states.



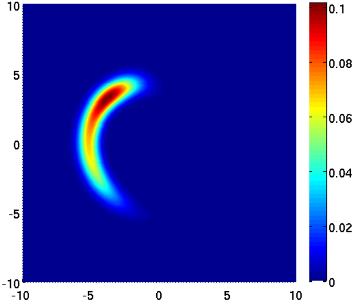
The dispersive regime (Single Quantum Trajectory IV)

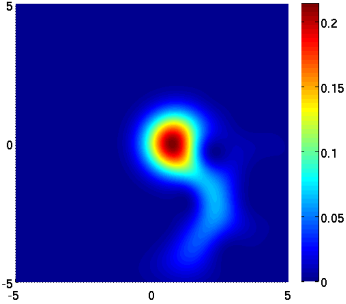
- The two states coexist during the transition —1st order transition.
- During switching there is a 'spiral rotation' of the two states in the phase diagram.



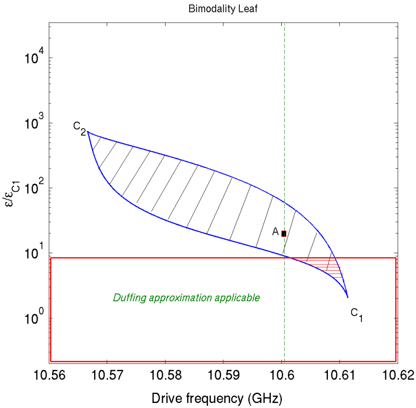
The dispersive regime (Single Quantum Trajectory V)

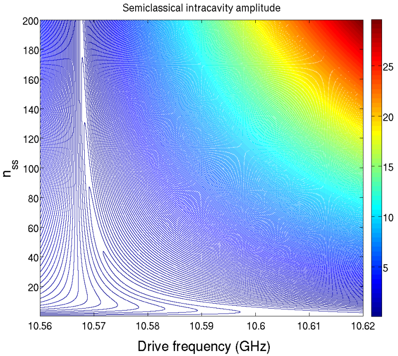
- The two states coexist during the transition —1st order transition.
- One of the three mean-field states is unstable in fluctuations (right).





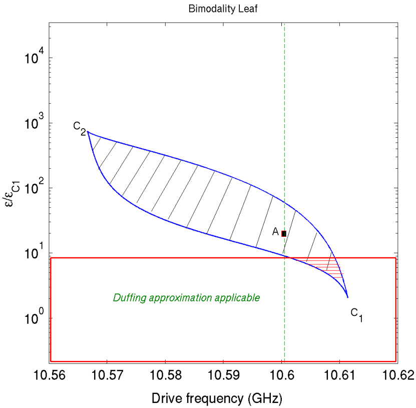
The point A in the phase space marks the 1st order phase transition.

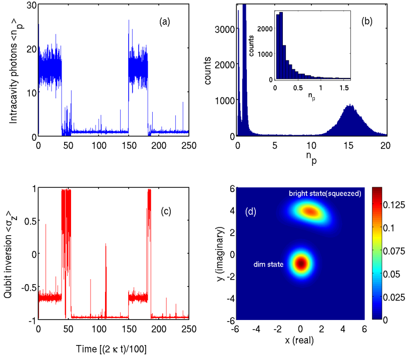




Concluding Graphs II

The point A in the phase space marks the 1st order phase transition.





Thank you for your attention!

Diffusive approximation and the Fokker-Planck Equation I

- We keep terms to second order in ε. For the deterministic evolution we have:

$$
\begin{aligned}
G(\psi) = & H\psi - \frac{i}{2}\gamma_0\{I + \varepsilon(C^\dagger + C) + \varepsilon^2 C^\dagger C\}\psi + \\
& \frac{i}{2}\gamma_0\{1 + \varepsilon\left\langle C^\dagger + C\right\rangle_\psi + \varepsilon^2\left\langle C^\dagger C\right\rangle_\psi\}\psi
\end{aligned} \tag{32}
$$

- and for the stochastic evolution:

$$
\begin{aligned}
W[\psi|\tilde{\psi}] = & \gamma_0\left(1 + \varepsilon\left\langle C^\dagger + C\right\rangle_{\tilde{\psi}} + \varepsilon^2\left\langle C^\dagger C\right\rangle_{\tilde{\psi}}\right) \\
& \delta\left[\frac{(I + \varepsilon C)\tilde{\psi}}{\sqrt{1 + \varepsilon\left\langle C^\dagger + C\right\rangle_{\tilde{\psi}} + \varepsilon^2\left\langle C^\dagger C\right\rangle_{\tilde{\psi}}}} - \psi\right] \\
= & \gamma_0\left(1 + \varepsilon\left\langle C^\dagger + C\right\rangle_{\tilde{\psi}} + \varepsilon^2\left\langle C^\dagger C\right\rangle_{\tilde{\psi}}\right)\delta\left[\tilde{\psi} - \psi + \varepsilon M(\tilde{\psi}) + \varepsilon^2 N(\tilde{\psi})\right],
\end{aligned} \tag{33}
$$

- The Liouville Master Equation is reduced to a Fokker-Planck equation for the probability density functional.

Diffusive approximation and the Fokker-Planck Equation II

- The non-linear drift operator $K(\psi)$ becomes

$$K(\psi) = H\psi + \frac{i}{2}\gamma_0\epsilon(C - C^\dagger)\psi + \\ + i\gamma_0\epsilon^2 \left\{ \frac{1}{2}\left\langle C + C^\dagger \right\rangle_\psi C - \frac{1}{8}\left\langle C + C^\dagger \right\rangle_\psi^2 - \frac{1}{2}C^\dagger C \right\} \psi. \tag{34}$$

- Non-vanishing and finite diffusive contribution requires $\gamma_0 = \epsilon^{-2}\gamma$, where γ is independent of ϵ. On the other hand, the drift operator $K(\psi)$ contains a term which is proportional to $\gamma_0\epsilon = \epsilon^{-1}\gamma$. To avoid divergence we impose that the operator C is self-adjoint.

$$K(\psi) = H\psi + i\gamma \left\{ \langle C \rangle_\psi C - \frac{1}{2}\langle C \rangle_\psi^2 - \frac{1}{2}C^2 \right\} \psi, \tag{35}$$

- The operator M assumes the form:

$$M(\psi) = (C - \langle C \rangle_\psi)\psi. \tag{36}$$

- The conditions leading to the diffusion limit depend on the number and the nature of the jump operators.

Diffusive approximation and the Fokker-Planck Equation III

- With these expressions, the Fokker-Planck equation is equivalent to the following stochastic Schroedinger equation in Itô form:

$$d\psi(t) = -iK(\psi(t))dt + \sqrt{\gamma}M(\psi(t))dW(t), \tag{37}$$

- $dW(t)$ is an increment of a real Wiener process.

- Numerical solution with various algorithms:
  - The Euler scheme. A scheme of order 1.
  - Combination of the 4th order Runge-Kutta scheme for the drift and the Euler scheme for the diffusion. A scheme of order 1.
  - The second-order weak scheme (Platen, 1992). A scheme of order 2.