Quiz 6
Math 250

For each of the following transformations $f : \mathbb{R}^2 \to \mathbb{R}^2$, either show that f is linear and write down its standard matrix, or explain why f is not linear.

(1) $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix}$

(2) $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 x_2 \\ x_1 \end{bmatrix}$

(1)

$$\begin{aligned} f\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}\right) &= \begin{bmatrix} x_1 + y_1 + x_2 + y_2 \\ x_1 + y_1 \end{bmatrix} \\ &= \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 \end{bmatrix} \\ &= f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + f\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned} f\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) &= \begin{bmatrix} cx_1 + cx_2 \\ cx_1 \end{bmatrix} \\ &= \begin{bmatrix} c(x_1 + x_2) \\ cx_1 \end{bmatrix} \\ &= cf\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \end{aligned}$$

Thus, f is linear. Its standard matrix A is given by

$$\begin{aligned} A &= \begin{bmatrix} f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

(2)

$$\begin{aligned} f\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) &= \begin{bmatrix} (cx_1)(cx_2) \\ cx_1 \end{bmatrix} \\ &= \begin{bmatrix} c^2(x_1 x_2) \\ cx_1 \end{bmatrix} \\ cf\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \begin{bmatrix} c(x_1 x_2) \\ cx_1 \end{bmatrix} \\ \Rightarrow f\left(\begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}\right) &\neq cf\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \end{aligned}$$

Thus, f is *not* linear.