Quiz 11
Math 250

Let $W = \text{Span } S$, where $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \right\}$, and let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(1) Find an *orthonormal* basis for W.
(2) Find the orthogonal projection of $\vec{u}$ on W.

(1) Use the Gram-Schmidt process to find an *orthogonal* basis $\{v_1, v_2\}$. We get that:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \vec{v_2} &= \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}. \end{aligned}$$

To get an *orthonormal* basis $\{\vec{w_1}, \vec{w_2}\}$, we set

$$\vec{w_1} = \frac{\vec{v_1}}{\|\vec{v_1}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w_2} = \frac{\vec{v_2}}{\|\vec{v_2}\|} = \frac{1}{\sqrt{18}} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}.$$

(The solution is continued on the next page.)

(2) We can solve this in two different ways:

Method 1: Given an orthonormal basis $\{\vec{w_1}, \vec{w_2}\}$ of W, the projection of $\vec{u}$ onto W is given by:

$$U_W(\vec{u}) = \vec{u} \cdot \vec{w_1} + \vec{u} \cdot \vec{w_2}.$$

This can be calculated using the given $\vec{u}$ and the set $\{\vec{w_1}, \vec{w_2}\}$ obtained in the previous part.

$$\begin{aligned} U_W(\vec{u}) &= \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{18}} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \right) \frac{1}{\sqrt{18}} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-3}{18} \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}. \end{aligned}$$

Method 2: The projection of $\vec{u}$ on W is also given by:

$$U_W(\vec{u}) = P_W u = C(C^T C)^{-1} C^T \vec{u}$$

where $C = \begin{bmatrix} 1 & 5 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$.

This method will certainly give you the correct answer, but is a lot more tedious in this case. In general, if you already have an orthogonal basis, it is much easier to use Method 1 instead.