

Workshop 5

1. Consider the following sequences:

$$a_n = \left(1 + \frac{1}{n}\right)^n ; \quad b_n = \left(1 + \frac{1}{n^2}\right)^n ; \quad c_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n .$$

a) Use your calculator to plot the first ten terms of each of these sequences. Then use this information to guess the limiting behavior of each of the sequences.

b) Replace n by x and use L'Hopital's Rule to find the limit of each as x tends to infinity.

2. Each of the following sequences has limit 0:

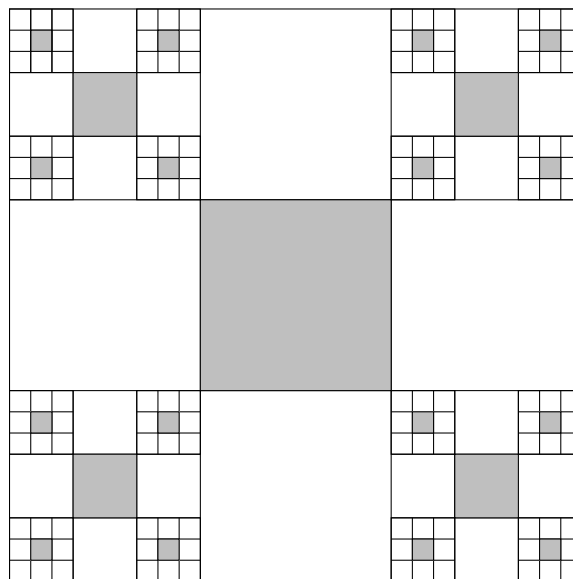
$$\left\{ \frac{1}{\sqrt{n}} \right\}_{n=1}^{\infty} \quad \left\{ \frac{1}{n} \right\}_{n=1}^{\infty} \quad \left\{ \frac{1}{n^2} \right\}_{n=1}^{\infty} \quad \left\{ \frac{1}{10^n} \right\}_{n=1}^{\infty}$$

a) For each sequence, state exactly how large n must be to ensure that the term a_n of the sequence (and all later terms as n increases) satisfy $|a_n| < 10^{-4}$.

b) Similarly, how large must n be to ensure that $|a_n| < 10^{-8}$?

c) Use this information to explain which sequence approaches 0 most rapidly and which approaches 0 least rapidly.

3. A 1×1 square is “dissected” by three equally spaced horizontal lines and by three equally spaced vertical lines. The central square is shaded. Then the bordering North-east, Northwest, Southeast, and Southwest squares are similarly dissected, with the central square shaded. Each of *those* dissected squares has a similar process done to their borders, etc. The diagram to the right shows this only for the first three steps but it is supposed to continue indefinitely.



a) How many new shaded squares are introduced at the n^{th} step? (There is one shaded square at the first step.) What is the side length of the squares which are introduced at the n^{th} step?

b) What is the sum, as n goes from 1 to ∞ , of the shaded area (all the shaded squares)? What is the sum, as n goes from 1 to ∞ , of the perimeters of all the shaded squares?

4. a) Two students are sharing a loaf of bread. Student Alpha eats half of the loaf, then student Beta eats half of what remains, then Alpha eats half of what remains, and so on. How much of the loaf will each student eat?

b) Two students are sharing a loaf of bread. Student Alpha, now hungrier and more ferocious, eats two-thirds of the loaf, then student Beta eats half of what remains, then Alpha eats two-thirds of what remains, then Beta eats half of what remains, and so on. How much of the loaf will each student eat?

c) Now start with three students: Alpha, Beta, and Gamma. They decide to share a loaf of bread. Alpha eats half of the loaf, passes what remains to Beta who eats half, and then on to Gamma who eats half, and then back to Alpha who eats half, and so on. How much of the loaf will each student eat?