Work the problems without a calculator, but use a calculator to check results. And try differentiating your answers in part III as a useful check.

I. Applications of Integration

Find the volumes of the solids obtained by rotating the indicated region $\mathcal{R}$ in the xy-plane about the specified axis:

#1. We sketch the first region.

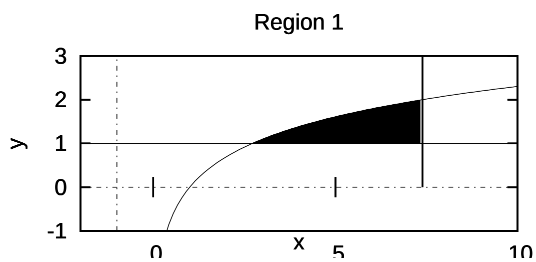

Figure 1: $1 \le y \le \ln(x)$, $x \le e^2$

Find the corners at $(e, 1)$, $(e^2, 1)$, $(e^2, 2)$ and label them as well.

Part (a) Rotate about the axis $y = -1$.
The cross-sections are washers (annuli). The integral is

$$\int_e^{e^2} \pi(1 + \ln x)^2 - \pi(2)^2\, dx$$

Expand as $\pi \int_e^{e^2} (\ln x)^2\, dx + 2\pi \int_e^{e^2} \ln x\, dx - 3\pi \int_e^{e^2} dx$ and as you do the first integral by parts, observe there is cancellation with the second integral.
Answer: $\pi e^2 + 2\pi e$.
On the calculator, this is about 40.3, which is what `fnInt(y1,X,e,e`^{2}`)` will give you if $y1$ is set to be $\pi(1 + \ln x)^2 - 4\pi$

(b) Rotate about the axis $x = -2$.
Using shells we get

$$\int_e^{e^2} 2\pi(x+2)(\ln x - 1)$$

This breaks into two integrals with the first done by parts:

$$2\pi \int_e^{e^2} (x+2)\ln x \, dx - 2\pi \int_e^{e^2} (x+2)\, dx$$

and eventually simplifies to $(\pi/2)(e^4 + e^2 + 8e)$ or about 131.

But it's much simpler integrated along the y-axis using washers:

$$\int_1^2 \pi(2+e^2)^2 - \pi((2+e^y)^2 \, dy$$

This expands to $\pi \int_1^2 4e^2 + e^4 - 4e^y - e^{2y} \, dy$ which can be integrated directly.

#2 We sketch the second region.

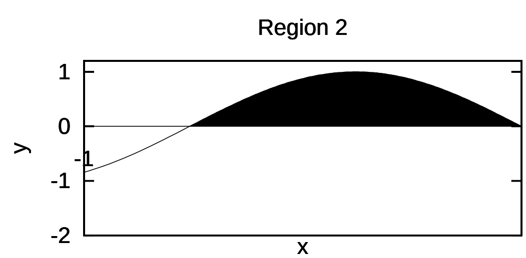

Figure 2: $1 \le y \le \ln(x)$, $x \le e^2$

(a) Rotating about the axis $y = -2$ we have the integral

$$\int_0^\pi \pi(2+\sin x)^2 - \pi(2)^2 \, dx = \pi \int_0^\pi 4\sin x + \sin^2 x \, dx$$

which comes out to $8\pi + \pi^2/2$ (about 30).

(b) Rotating about the axis line $x = -1$ and using shells, we have

$$\int_0^\pi 2\pi(1+x)\sin(x)\, dx$$

which can be done by parts as it stands; but it's natural to first expand $(1+x)\sin x = \sin x + x \sin x$ and then break the problem into two pieces. This comes out to $4\pi + 2\pi^2$ or about 32.

#3 We sketch the third region—this is a little awkward on the graphing calculator, and the simplest approach is to use the "parametric curves" option.

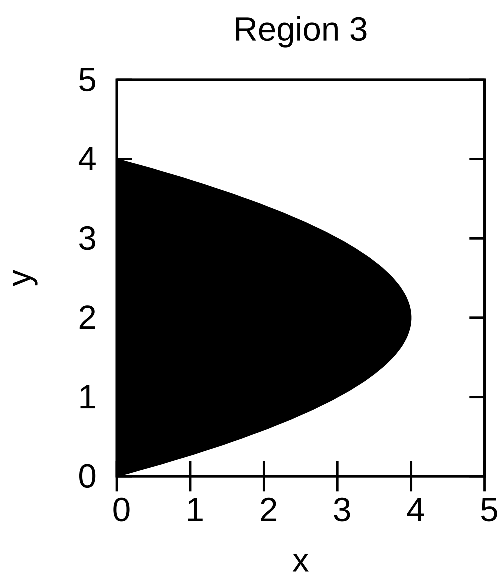

Figure 3: Between the y-axis and $x = y(4-y)$

As always, we should first identify the corners of the region, using $y(4-y) = 0$, and label them in the picture.
Rotating this region about the y-axis, we will want to integrate along the y-axis and work with discs:

$$\int_0^4 \pi[y(4-y)]^2\,dy$$

Expanding the integrand as $16y^2 - 8y^3 + y^4$) we have a polynomial to integrate, and the result is $\pi(16 \cdot 4^3/3 - 8(4^4/4) + 4^5/5)$ which can certainly be simplified further (to $512\pi/15$), but in a test situation we are not very interested in this kind of arithmetic. If you check the answer on your calculator you can use `fnInt` with either X or Y as the variable of integration.

#4 We compute the work done against gravity building a pyramid with a square base, 800 feet on a side, 500 feet tall, and with density 170 lbs/sq. foot. Slicing this into horizontal layers, we can measure the height x of the layer either from ground level or from the top of the pyramid—and it's a little more convenient to measure from the top vertex.
The useful picture is a vertical triangular cross–section, to work out the dimensions of the horizontal cross–sections (squares).

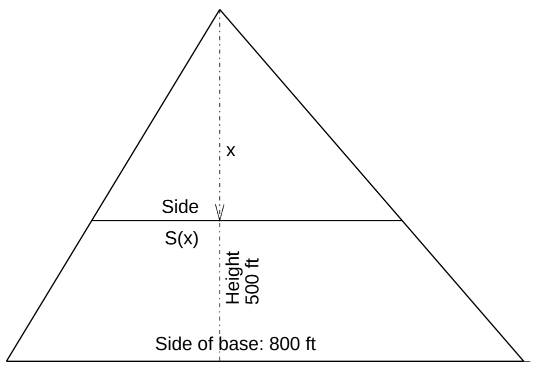

Figure 4: Pyramid (Vertical Section)

By similar triangles we see that the side $S(x)$ of the cross–section at depth x (measured from the top) satisfies $S(x)/800 = x/500$, $S(x) = (8/5)x$.
The mass of the cross section at depth x with thickness Δx is $(170)S(x)^2\Delta(x)$ and the work done lifting that section (to a height of $500 - x$) is

$$(500 - x) \cdot (170)S(x)^2\Delta x = (170)(8/5)^2(500 - x)x^2\Delta x$$

The total work is approximated by the sum

$$\sum_i (170)(8/5)^2(500 - x_i)x_i^2\Delta x$$

and in the limit this gives the integral

$$\int_0^{500} (170)(8/5)^2(500 - x)x^2\, dx$$

We were asked to write this integral down and justify it, but not to compute it.
If we decide instead to measure height from ground level the expression for the work will be similar:

$$\int_0^{500} (170)(8/5)^2 x(500 - x)^2\, dx$$

#5 (*a*) How do we know that there is a point in the interval $[5, 7]$ where the function $f(x) = (x^2 - 4)^{-1/2}$ takes on its average value over that interval? By the Mean Value Theorem for Integrals, which requires us to check that the function is continuous on $[5, 7]$. As $\sqrt{x^2 - 4}$ is continuous, and nonzero, on that interval, its reciprocal is continuous as well. This justifies our claim.

(*b*) To calculate the point in question, we first work out the average, using the substitution $x = 2\sec\theta$. This brings us to

$$\frac{1}{7-5}\int_5^7 \frac{dx}{\sqrt{x^2-4}} = \frac{1}{2}\ln\left(x + \sqrt{x^2-4}\right)|_5^7$$

so that the average $\bar{f}$ is $\frac{1}{2}\ln\left(\frac{7+\sqrt{45}}{5+\sqrt{21}}\right)$.
The point x we are looking for satisfies

$$\frac{1}{\sqrt{x^2-4}} = (1/2)\ln\left(\frac{7+\sqrt{45}}{5+\sqrt{21}}\right)$$

which becomes

$$x = \sqrt{4 + \frac{4}{\left(\ln\left(\frac{7+\sqrt{45}}{5+\sqrt{21}}\right)^2\right)}}$$

This turns out to be about 5.9, in other words well inside the expected interval.

II. Numerical Methods

#6 How many subintervals of $[0, 2]$ should we use to ensure an accuracy within 10^{-6} when we approximate $\int_0^2 4x^3 - x^4\,dx$ using the Midpoint Rule or Simpson's Rule?
We first need estimates for the 2nd and 4th derivatives of our function

$$f''(x) = 24x - 12x^2$$

$$f^{(iv)}(x) = 24$$

Since the 4th derivative is constant we will take $K_4 = 24$. For the 2nd derivative, either we use the fact that it is a parabola with zeroes at $x = 0$ and $x = 2$, or we use the methods of calculus, to find that the maximum value of f'' on $[0, 2]$ occurs at $x = 1$ and thus $K_2 = 12$.
Our error estimates become

$$\text{Error}(M_N) \leq 12(2)^3/24N^2$$

$$\text{Error}(S_N) \leq 24(2)^5/180N^4$$

So using the Midpoint rule, we require $4/N^2 \leq 10^{-6}$ which translates into $N > \sqrt{4 \cdot 10^6} = 2 \cdot 10^3 = 2000$.
Using Simpson's rule, we require $24(32)/180N^4 \leq 10^{-6}$ or $N \geq \sqrt[4]{\frac{(24)(32)10^6}{180}} = 10\sqrt[4]{24 \cdot 3200/180}$ and the arithmetic is unpleasant; a calculator will tell us that $N = 50$ will do here.
Running the numerical integration programs on my calculator with `N=25` (which for this program, gives me Simpson's rule with 50 subintervals), I get the predicted accuracy—and not much more.
With $N = 2000$ my version of the program crashed, but another version gave 9.599999333 for the Midpoint Rule, with the predicted accuracy.

#7 If the Trapezoidal Rule with 30 subintervals is used to estimate $\int_1^4 f(x)\,dx$, giving the estimate 3.14286, and $f''(x)$ has the graph shown, give a range of values guaranteed to include the true value of the integral.

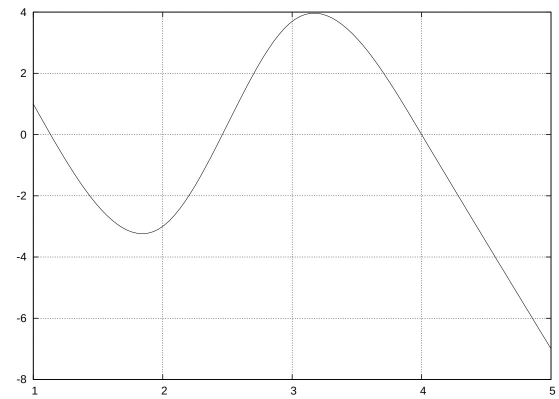

Figure 5: f''

From the graph, K_2 can be taken to be 4 on the interval $[1, 4]$. So our error estimate is $4{\cdot}3^3/12{\cdot}900 = 1/100$. Therefore the true value lies in the interval $[3.13, 3.16]$, or if you prefer, $[3.132, 3.153]$—any more precision would be pointless. Maybe it's π, or $22/7$, or 3.15, or something else ...

III. Techniques of Integration

#8 Four trigonometric integrals:

The first three succumb to the normal substitutions $u = \cos x$, $u = \tan x$, or $u = \sec x$, the last one needs a reduction formula (obtained by integration by parts).

(a) $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$ $\quad$ (b) $\tan x + \frac{1}{3}\tan^3 x + C$

(c) $\frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5(x) + \frac{1}{3}\sec^3(x) + C$ $\quad$ (d) $\frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

#9 A mix of integration by parts and other things.

(a): by parts. $(1/6)x^6(\ln x)^2 - (1/18)x^6 \ln x + (1/108)x^6 + C$

(b): Just a substitution! $\ln(\ln x) + C$.

(c): First substitute $u = \sqrt{x}$, then integrate by parts.

$2\sqrt{x}\sin(\sqrt{x}) + 2\cos(\sqrt{x}) + C$

(d) Integration by parts to reduce to $\int \frac{x^3}{1+x^2}\,dx$, then use $\frac{x^3}{(1+x^2)} = x - \frac{x}{1+x^2}$ to reduce to $\int \frac{x}{1+x^2}\,dx$, and a substitution will handle that.

$(x^3/3)\tan^{-1}(x) - x^2/6 + (1/6)\ln(1+x^2) + C$

(e) Similar, but after reducing to $\int \frac{dx}{x\sqrt{1-x^2}}$, $x = \sin\theta$ is needed.

$-\frac{\sin^{-1} x}{x} + \ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) + C$

(f) First $u = \sqrt{x}$, then integration by parts.

$2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$
(g) The function $\frac{1}{1+e^x} - \frac{1}{2}$ is an odd function. The function $\cos(x)$ is an even function. Therefore, the function $\left(\frac{1}{1+e^x} - \frac{1}{2}\right)\cos(x)$ is an odd function. We conclude

$$\int_{-\pi/2}^{\pi/2} \left(\frac{1}{1+e^x} - \frac{1}{2}\right)\cos(x)\,dx = 0.$$

Rewriting this, we obtain the solution to the problem.

#10 Integrals that need either a trigonometric substitution (possibly completing the square first) or the method of partial fractions.
(a) $x = 5\tan\theta$. $(1/250)(\tan^{-1}(x/5) + \frac{x}{50(x^2+25)}) + C$
(b) Partial fractions, of the form $\frac{Ax+b}{x^2+e6} + \frac{C}{x+1}$, leading to $A = 1/37$, $B = 36/37$, $C = -1/37$.
$(1/37)((1/2)\ln(x^2+36) - \ln(x+1) + 6\tan^{-1}(x/6)) + C$. 4 (c) Completing the square, $\int \frac{dx}{\sqrt{1-(x-1)^2}}$, so substitute $u = x - 1$.
Ans: $\sin^{-1}(x-1) + C$
(d) Partial fractions, in the form $\frac{Ax+B}{(x+3)^2} + \frac{C}{x-5}$
$A = 5/8$, $B = -1/8$, $C = 3/8$.
$(5/8)\ln(x+3) + (3/8)\ln(x-5) + 2/(x+3) + C$
(e) $x = 4\sin\theta$
$\frac{x}{\sqrt{16-x^2}} - \sin^{-1}(x/4) + C$
(f) Complete the square, $x^2 + 4x + 9 = (x+2)^2 + 5$, $u = x + 2$
$(1/\sqrt{5})\tan^{-1}(\frac{x+2}{\sqrt{5}}) + C$

#11 Evaluate $\int \sin(\ln x)\,dx$ using two integrations by parts. Would another method work?

$$\begin{aligned}
\int \sin(\ln x)\,dx &= x\sin(\ln x) - \int \cos(\ln x)\,dx \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln(x))\,dx \\
\int \sin(\ln x)\,dx &= (x/2)(\sin(\ln x) - \cos(\ln x)) + C
\end{aligned}$$

Another method: the natural substitution $u = \ln(x)$ converts this to $\int e^u \sin(u)\,du$, a more standard form, that still requires the same technique.

#12 Four improper integrals:

(a) The partial fractions expansion $\dfrac{1}{(x-3)(x-4)} = \dfrac{1}{x-4} - \dfrac{1}{x-3}$ gives

$$\int_5^\infty \frac{dx}{(x-3)(x-4)} = \lim_{b\to\infty} \left(\ln|x-4| - \ln|x-3|\right)\Big|_5^b = \lim_{b\to\infty} \ln\left(\frac{b-4}{b-3}\right) + \ln 2 = \ln 2.$$

(b) L'Hôpital's Rule lets us write

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = \lim_{x\to 0+} -x = 0.$$

This can be rewritten $\lim\limits_{a\to 0+} a\ln a = 0$. Now we get

$$\int_0^1 \ln x\,dx = \lim_{a\to 0+} \int_a^1 \ln x\,dx = \lim_{a\to 0+} (x\ln x - x)\Big|_a^1 = -1.$$

(c) Integration by parts gives $\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx$. Now the fact $\lim\limits_{b\to\infty} b^n e^{-b} = 0$ (which we get from l'Hôpital's Rule) lets us write

$$\int_0^\infty x^n e^{-x}\,dx = n\int_0^\infty x^{n-1}e^{-x}\,dx.$$

Therefore, $\int_0^\infty x^3 e^{-x}\,dx = 3\int_0^\infty x^2 e^{-x}\,dx = 6\int_0^\infty x^1 e^{-x}\,dx = 6\int_0^\infty e^{-x}\,dx =$ 6.

(d) The substitution $x = 3\tan\theta$ gives

$$\int \frac{dx}{9+x^2} = \int \frac{3\sec^2\theta\,d\theta}{9(1+\tan^2\theta)} = \int \frac{d\theta}{3} = \frac{\theta}{3} + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Therefore, $\int_{-\infty}^\infty \dfrac{dx}{9+x^2} = \lim\limits_{b\to\infty}\left(\dfrac{1}{3}\tan^{-1}\left(\dfrac{b}{3}\right) - \dfrac{1}{3}\tan^{-1}\left(\dfrac{-b}{3}\right)\right) = \dfrac{1}{3}\left(\dfrac{\pi}{2} - \left(-\dfrac{\pi}{2}\right)\right) = \dfrac{\pi}{3}$.

#13 The convergence and divergence can be determined using the Comparison Test.

(a) For $x \geq 7$ we know $0 < x - |\cos x| \leq x$, hence $\dfrac{1}{x-|\cos x|} \geq \dfrac{1}{x} > 0$. The divergence of $\int_7^\infty \dfrac{dx}{x}$ implies the divergence of $\int_7^\infty \dfrac{dx}{x-|\cos x|}$.

(b) For $x \geq 5$ we know $0 < \dfrac{1}{e^{x^2}} < \dfrac{1}{e^x}$. The convergence of $\int_5^\infty \dfrac{dx}{e^x}$ (which is just $\int_5^\infty e^{-x}\,dx$) implies the convergence of $\int_5^\infty \dfrac{dx}{e^{x^2}}$.