Math 116: Symmetry and Shape
Euclid's Elements

The Elements:

Euclid's thirteen volume book *The Elements* was written around 300 BC, and is the most famous mathematical text of all time. Almost of all the geometry that we learn in school can be found in these books.

The elements started with 23 definitions, five postulates, and five "common notions," and then systematically built the rest of plane and solid geometry upon this foundation.

1. Postulates:

   Postulate 1. To draw a straight line from any point to any point.
   Postulate 2. To produce a finite straight line continuously in a straight line.
   Postulate 3. To describe a circle with any center and radius.
   Postulate 4. That all right angles equal one another.
   Postulate 5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

   The first and second postulates give us the geometric tool known as the *straightedge*. We can think of this as a ruler, but without markings that keep track of length.

   The third postulate gives us a the geometric tool known as the *compass*. This allows us to construct circles given a specified center and radius.

2. Some propositions (from Book I) that are constructions:

   Proposition 9. To bisect a given rectilinear angle.
   Proposition 10. To bisect a given finite straight line.
   Proposition 11. To draw a straight line at right angles to a given straight line from a given point on it.
   Proposition 12. To draw a straight line perpendicular to a given infinite straight line from a given point not on it.
   Proposition 31. To draw a straight line through a given point parallel to a given straight line.

3. Some constructible regular polygons:

   - Equilateral Triangle (I.1)
   - Square (I.46)
   - Regular Pentagon (IV.11)
   - Regular Hexagon (IV.15)
   - Regular Pentakaidecagon (IV.16)

Gauss's Theorem: A regular n-gon can be constructed with straightedge and compass alone if all the odd prime factors of n are distinct *Fermat primes*.

This means that all the *odd* prime factors of n must be different (so that there are no repeats), and they must each be of the form $F_k = 2^{2^k} + 1$. There is no restriction on how many times 2 appears as a factor of n.

Primes of the form $2^{2^k} + 1$ are called Fermat primes. The known Fermat primes are:

$$
\begin{aligned}
2^{2^0} + 1 &= 3 \\
2^{2^1} + 1 &= 5 \\
2^{2^2} + 1 &= 17 \\
2^{2^3} + 1 &= 257 \\
2^{2^4} + 1 &= 65537
\end{aligned}
$$