LTCC: Representation Theory of Finite Groups
Exercise Set 5

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Prove part (ii) of Clifford's Theorem: Suppose $H \lhd G$, χ is an irreducible character of G, $\psi_1, \dots \psi_m$ the constituents of $\chi \downarrow H$. Then $\langle \chi \downarrow H, \psi_i \rangle$ is the same for all ψ_i.

   Solution: Let V be the $\mathbb{C}[G]$-module with character χ, U_i the $\mathbb{C}[H]$-module with character ψ_i, and suppose $\langle \chi \downarrow H, \psi_1 \rangle = d$. Then $V \downarrow H$ has a $\mathbb{C}[H]$-submodule X_1 such that X_1 is the direct sum of d copies of U_1. We have seen in the proof of part (i) of Clifford's Theorem that $U_i = gU_1$ for some $g \in G$, so gX_1 must decompose into d copies of U_i, and thus $\langle \chi \downarrow H, \psi_i \rangle = d$ for all i.

2. Suppose H is a subgroup of G of index 2 (hence, normal).

   (a) Let χ be a character of G. Show that $\chi \downarrow H$ is either irreducible or is the sum of two distinct irreducible characters of H of the same degree.

   Solution: Let $\chi \downarrow H = d_1\psi_1 + \cdots + d_k\psi_k$ where $\psi_1, \psi_2, \dots, \psi_k$ are the irreducible of characters of H. Then $\sum_i d_i^2 \leq 2$, so either $\chi \downarrow H = \psi_i$ or $\chi \downarrow H = \psi_i + \psi_j$ for some $i \neq j$. In the latter case, we know ψ_i and ψ_j have the same degree by Clifford's Theorem.

   (b) Show that $\chi \downarrow H$ is irreducible iff $\chi(g) \neq 0$ for some $g \in G - H$.

   Solution: As before, let $\chi \downarrow H = d_1\psi_1 + \cdots + d_k\psi_k$ where $\psi_1, \psi_2, \dots, \psi_k$ are the irreducible of characters of H. By Prop 2 of Section 5.1, we know $\sum_i d_i^2 < 2$ iff $\chi(g) \neq 0$ for some $g \in G - H$. Thus, $\chi(g) \neq 0$ for some $g \in G - H$ iff $\sum_i d_i^2 = 1$, in which case $\chi \downarrow H = \psi_i$ for some i (and hence is irreducible).

   (c) Show that G has a linear character λ such that

   $$\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H \end{cases}$$

   Solution: We observe that $G/H \cong C_2$ in this case, and C_2 has a character ξ such that $\xi(e) = 1$ and $\xi(x) = -1$ for $x \neq e$. The lift λ of this character is given by $\lambda(g) = \xi(gH)$, and hence $\lambda(g) = 1$ if $g \in H$ and $\lambda(g) = -1$ if $g \notin H$.

   (d) Show that $\chi \downarrow H$ is irreducible iff χ and $\chi\lambda$ are not equal characters.

   Solution: We have seen in part (b) that $\chi \downarrow H$ is irreducible iff $\chi(g) \neq 0$ for some $g \in G - H$. In this case, $(\chi\lambda)(g) = \chi(g)\lambda(g) = -\chi(g) \neq \chi(g)$.

3. Given that the complete list of degrees of the irreducible characters of S_7 is

$$1, 1, 6, 6, 14, 14, 14, 14, 15, 15, 20, 21, 21, 35, 35,$$

and that A_7 has nine conjugacy classes, find the complete list of degrees of the irreducible characters of A_7.

Solution: Since $[S_7 : A_7] = 2$, we know that each character of S_7 either restricts to an irreducible character of A_7 or is the sum of two irreducible characters of equal degree, and by part (d) of the previous exercise, we know $\chi \downarrow A_7$ is irreducible iff χ and $\chi\lambda$ are not equal characters (where λ is defined in part (c)), and these characters would have the same degree and have the same restriction to A_7. Conversely, characters of S_7 which are unique of some fixed degree must restrict to the sum of two distinct irreducible characters of A_7.

Since 20 occurs only once in the list of degrees for S_7, the restriction of the irreducible character of degree 20 to A_7 must be the sum of two different irreducible characters of degree 10. From the remaining fourteen irreducible characters of S_7, after restricting to A_7 we get at least seven irreducible characters of A_7, and we get precisely seven if and only if the restriction of each of the fourteen characters is irreducible. Since A_7 has exactly nine conjugacy classes, we must in fact have only these seven irreducible characters (in addition to the two we have already obtained of degree 10). Thus, the list of degrees of the irreducible characters of A_7 are:

$$1, 6, 14, 14, 15, 10, 10, 21, 35.$$

4. Let $G = S_4$ and let H be the subgroup $\langle(1\ 2\ 3)\rangle \cong C_3$. Calculate the induced characters $\psi_i \uparrow G$ for each irreducible character ψ_i of H. Decompose each of these as a sum of the irreducible characters χ_i of G. (You may use the tables given in lecture.)

Solution: Since $\psi \uparrow G(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(y^{-1}gy)$, we have $\psi \uparrow G(e) = \frac{|G|}{|H|}\psi(e)$, so $\psi_i \uparrow G(e) = 8$ for all i (since the characters of C_3 are all linear).

Next, we observe that the elements in S_7 conjugate to $(1\ 2\ 3)$ are the other 3-cycles, and in particular, for $y \in S_7$, we have $y(1\ 2\ 3)y^{-1} = (y(1)\ y(2)\ y(3))$. Therefore, $y(1\ 2\ 3)y^{-1} \in H$ iff $y(4) = 4$, so there are 6 elements that satisfy this (since $|S_3| = 6$).

We therefore compute:

$$\psi_1 \uparrow G(1\ 2\ 3) = \frac{1}{3}\sum_{y\in G} \dot{\psi}_1(y^{-1}(1\ 2\ 3)y) = \frac{1}{3}6 = 2$$
$$\psi_2 \uparrow G(1\ 2\ 3) = \frac{1}{3}\sum_{y\in G} \dot{\psi}_2(y^{-1}(1\ 2\ 3)y) = \frac{1}{3}(3\omega + 3\omega^2) = -1$$
$$\psi_3 \uparrow G(1\ 2\ 3) = \frac{1}{3}\sum_{y\in G} \dot{\psi}_3(y^{-1}(1\ 2\ 3)y) = \frac{1}{3}(3\omega^2 + 3\omega) = -1.$$

For any nonidentity element $g \in S_4$ such that g is not a 3-cycle, we have

$$\psi_i \uparrow G(g) = 0.$$

Therefore, by taking inner products with the irreducible characters of S_4, we have:

$$\psi_1 \uparrow G = \chi_1 + \chi_2 + \chi_4 + \chi_5$$
$$\psi_2 \uparrow G = \chi_3 + \chi_4 + \chi_5$$
$$\psi_3 \uparrow G = \chi_3 + \chi_4 + \chi_5.$$

5. Suppose that H is a subgroup of G, and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Let ψ be an irreducible character of H. Given $\psi \uparrow G = d_1\chi_1 + \cdots + d_k\chi_k$, show that
$$\sum_i d_i^2 \leq [G : H].$$

   Solution: We observe $d_i = \langle \psi \uparrow G, \chi_i \rangle = \langle \psi, \chi_i \downarrow H \rangle$, by the Frobenius Reciprocity Theorem. Hence, since ψ is irreducible, $\chi_i \downarrow G = d_i\phi + \xi$ where either ξ is a character of H or $\xi = 0$. Therefore, $\chi_i(e) \geq d_i\psi(e)$.

   Since $[G : H]\psi(e) = d_1\chi_1(e) + \cdots + d_k\chi_k(e)$, we have
$$[G : H]\psi(e) \geq d_1^2\psi(e) + \cdots + d_k^2\psi(e),$$
   and hence the result follows.

6. Suppose H is a subgroup of G of index 2 (hence, normal). Let ψ be am irreducible character of H. Show that $\psi \uparrow G$ is either irreducible or is the sum of two distinct irreducible characters of G of the same degree.

   Solution: Let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Given $\psi \uparrow G = d_1\chi_1 + \cdots + d_k\chi_k$, we know from the previous exercise that
$$\sum_i d_i^2 \leq [G : H] = 2.$$

Therefore, $\psi \uparrow G = \chi_i$ or $\psi \uparrow G = \chi_i + \chi_j$ for some $i \neq j$.

In the latter case, we have $2\psi(e) = \chi_i(e) + \chi_j(e)$. Moreover, by Frobenius Reciprocity, we have $\langle \psi, \chi_i \downarrow H \rangle = \langle \psi \uparrow G, \chi_i \rangle = 1$, so by Q2a, we have $\chi_i(e) = \psi(e)$ or $\chi_i(e) = 2\psi(e)$. The same argument applies to χ_j as well as χ_i, and hence we must have $\chi_i(e) = \psi(e) = \chi_j(e)$. Therefore, χ_i and χ_j have the same degree in this case.

7. Let $G = S_4$. Find the characters ϕ^λ of all the permutation modules M^λ.

   Solution: The permutations of 4 are:

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

   Therefore, the characters (obtained by finding the number of tabloids of indexing partition shape λ fixed by the given element g_i) are:

g_i :	e	(1 2)	(1 2)(3 4)	(1 2 3)	(1 2 3 4)
$\phi^{(4)}$	1	1	1	1	1
$\phi^{(3,1)}$	4	2	0	1	0
$\phi^{(2,2)}$	6	2	2	0	0
$\phi^{(2,1,1)}$	12	2	0	0	0
$\phi^{(1,1,1,1)}$	24	0	0	0	0

   The following is not required for this exercise, but observe that, using the numbering in the character table of S_4 from lecture, we have:

$$\begin{aligned}
\phi^{(4)} &= \chi_1 \\
\phi^{(3,1)} &= \chi_4 + \chi_1 \\
\phi^{(2,2)} &= \chi_3 + \chi_4 + \chi_1 \\
\phi^{(2,1,1)} &= \chi_5 + \chi_3 + 2\chi_4 + \chi_1 \\
\phi^{(1,1,1,1)} &= \chi_2 + 2\chi_3 + 3\chi_4 + \chi_1.
\end{aligned}$$

8. The permutation modules M^μ decompose as a direct sum of Specht modules S^λ with $\lambda \leq \mu$ in reverse lexicographic order. In fact, we have a nice combinatorial way

of computing the multiplicities of the S^λ in this decomposition: the multiplicity of S^λ in M^μ is given by the Kostka numbers $K_{\lambda\mu}$, which equal the number of tableau of shape λ filled with μ_1 copies of 1, μ_2 copies of 2, etc, with the filling weakly increasing across rows (each entry is greater than or equal to the one to its left) and strictly increasing down columns (each entry is strictly greater than the one above it). [We call these *semistandard tableau* of shape λ with *content* μ.]

Decompose the following modules into a direct sum of Specht modules.

(a) $M^{(2,1)}$

Solution: We have the following semistandard tableau satisfying the given conditions:

$$\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}.$$

Therefore, $M^{(2,1)} = S^{(3)} \oplus S^{(2,1)}$.

(b) $M^{(1,1,1,1)}$

Solution: We have the following semistandard tableau satisfying the given conditions:

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}.$$

Therefore, $M^{(1,1,1,1)} = S^{(4)} \oplus 3 \cdot S^{(3,1)} \oplus 2 \cdot S^{(2,2)} \oplus 3 \cdot S^{(2,1,1)} \oplus S^{(1,1,1,1)}$.

(c) $M^{(2,2,1)}$

Solution: We have the following semistandard tableau satisfying the given conditions:

$$\begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&3\\\hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 2&2\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3\\\cline{1-1}\end{array}$$

Therefore, $M^{(2,2,1)} = S^{(5)} \oplus 2 \cdot S^{(4,1)} \oplus 2 \cdot S^{(3,2)} \oplus S^{(2,2,1)}$.