LTCC: Representation Theory of Finite Groups
Exercise Set 5

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Prove part (ii) of Clifford's Theorem: Suppose $H \lhd G$, χ is an irreducible character of G, $\psi_1, \ldots \psi_m$ the constituents of $\chi \downarrow H$. Then $\langle \chi \downarrow H, \psi_i \rangle$ is the same for all ψ_i.

2. Suppose H is a subgroup of G of index 2 (hence, normal).

   (a) Let χ be a character of G. Show that $\chi \downarrow H$ is either irreducible or is the sum of two distinct irreducible characters of H of the same degree.

   (b) Show that $\chi \downarrow H$ is irreducible iff $\chi(g) \neq 0$ for some $g \in G - H$.

   (c) Show that G has a linear character λ such that

   $$\lambda(g) = \begin{cases} 1 & \text{if } g \in H \\ -1 & \text{if } g \notin H \end{cases}$$

   (d) Show that $\chi \downarrow H$ is irreducible iff χ and $\chi\lambda$ are not equal characters.

3. Given that the complete list of degrees of the irreducible characters of S_7 is

   $$1, 1, 6, 6, 14, 14, 14, 14, 15, 15, 20, 21, 21, 35, 35,$$

   and that A_7 has nine conjugacy classes, find the complete list of degrees of the irreducible characters of A_7.

4. Let $G = S_4$ and let H be the subgroup $\langle (1\ 2\ 3) \rangle \cong C_3$. Calculate the induced characters $\psi_i \uparrow G$ for each irreducible character ψ_i of H. Decompose each of these as a sum of the irreducible characters χ_i of G. (You may use the tables given in lecture.)

5. Suppose that H is a subgroup of G, and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Let ψ be an irreducible character of H. Given $\psi \uparrow G = d_1\chi_1 + \cdots + d_k\chi_k$, show that

   $$\sum_i d_i^2 \leq [G : H].$$

6. Suppose H is a subgroup of G of index 2 (hence, normal). Let ψ be an irreducible character of H. Show that $\psi \uparrow G$ is either irreducible or is the sum of two distinct irreducible characters of G of the same degree.

7. Let $G = S_4$. Find the characters ϕ^λ of all the permutation modules M^λ.

8. The permutation modules M^μ decompose as a direct sum of Specht modules S^λ with $\lambda \leq \mu$ in reverse lexicographic order. In fact, we have a nice combinatorial way of computing the multiplicities of the S^λ in this decomposition: the multiplicity of S^λ in M^μ is given by the Kostka numbers $K_{\lambda\mu}$, which equal the number of tableau of shape λ filled with μ_1 copies of 1, μ_2 copies of 2, etc, with the filling weakly increasing across rows (each entry is greater than or equal to the one to its left) and strictly increasing down columns (each entry is strictly greater than the one above it). [We call these *semistandard tableau* of shape λ with *content* μ.]

   Decompose the following modules into a direct sum of Specht modules.

   (a) $M^{(2,1)}$

   (b) $M^{(1,1,1,1)}$

   (c) $M^{(2,2,1)}$