LTCC: Representation Theory of Finite Groups
Exercise Set 4

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Suppose χ is a character of G and λ is a linear character of G.

   (a) Show that the product $\lambda\chi$ (given by $\lambda\chi(g) = \lambda(g)\chi(g)$) is also a character of G.

   Solution: Let $\rho : G \to \mathrm{GL}(V)$ be the representation associated to χ, and note that the representation associated with λ can be taken to be λ itself (see Week 3, Q6). Since $\lambda(g)$ is a scalar for all $g \in G$, we have that $\sigma : G \to \mathrm{GL}(V)$ given by $\sigma(g) = \lambda(g)\rho(g)$ is a homomorphism, since

   $$\sigma(gh) = \lambda(gh)\rho(gh) = \lambda(g)\lambda(h)\rho(g)\rho(h) = \lambda(g)\rho(g)\lambda(h)\rho(h) = \sigma(g)\sigma(h).$$

   Furthermore, the trace of the matrix $\sigma(g)$ is $\lambda(g)\chi(g)$ (again, since $\lambda(g)$ is a scalar). Thus, $\lambda\chi$ is the character of σ.

   (b) Show that if χ is irreducible, then so is $\lambda\chi$.

   Solution: We observe

   $$\langle \lambda\chi, \lambda\chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\lambda(g)||\chi(g)|.$$

   However, since λ is a degree 1 character, we have that $\lambda(g)$ is a root of unity for each $g \in G$, and therefore $|\lambda(g) = 1$ for all $g \in G$. Thus,

   $$\langle \lambda\chi, \lambda\chi \rangle = \frac{1}{|G|} \sum_{g \in G} |\lambda(g)||\chi(g)| = \frac{1}{|G|} \sum_{g \in G} |\chi(g)| = \langle \chi, \chi \rangle .$$

   Since a character is irreducible if and only if its inner product with itself is 1, the result follows.

2. Let V and W be vector spaces. If $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W, then the *tensor product* $V \otimes W$ is the vector space with basis $\{v_i \otimes w_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. [Note that for $v \in V$ and $w \in W$, we have $v \otimes w = (\sum_i \lambda_i v_i) \otimes (\sum_j \lambda_i v_i) = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$.] If V and W are in fact $\mathbb{C}[G]$-modules, we can define an action of G on $V \otimes W$ by $g \cdot (v \otimes w) = gv \otimes gw$ and extending linearly.

(a) Show that if the characters of V and W and χ and ψ, respectively, then the character of $V \otimes W$ is $\chi\psi$. [This shows that the product of any two characters of G is again a character of G. Note that this gives us an alternative proof of Exercise 1a, but Exercise 1a can also be solved more directly.]

Solution: Fix $g \in G$. We can find a basis $\{v_1, \ldots, v_n\}$ of V and $\{w_1, \ldots, w_m\}$ of W consisting of eigenvectors of g (by the proof of Section 3.1, Prop 3). Suppose $gv_i = \lambda_i v_i$ and $gw_j = \mu_j w_j$. Then $\chi(g) = \sum_i \lambda_i$ and $\psi(g) = \sum_j \mu_j$. Therefore, since $\{v_i \otimes w_j\}$ is a basis for $V \otimes W$, its character ϕ satisfies

$$\phi(g) = \sum_{ij} \lambda_i \mu_j = \sum_i \lambda_i \sum_j \mu_j = \chi(g)\psi(g).$$

(b) Let V be a $\mathbb{C}[G]$-module with basis $\{v_1, \ldots, v_n\}$, and let $\varphi : V \otimes V \to V \otimes V$ be the map given by $\varphi(v_i \otimes v_j) = v_j \otimes v_i$. Show that $\mathrm{Sym}(V) = \{x \in V \otimes V \mid \varphi(x) = x\}$ and $\mathrm{Alt}(V) = \{x \in V \otimes V \mid \varphi(x) = -x\}$ are complementary submodules of $V \otimes V$.

Solution: For any $g \in G$, we have $g\varphi(v_i \otimes v_j) = g(v_j \otimes v_i) = gv_j \otimes gv_i = \varphi(gv_i \otimes gv_j) = \varphi(g(v_i \otimes v_j))$. Thus, φ commutes with the action of G, and hence is a $\mathbb{C}[G]$-homomorphism.

Therefore, if $x \in \mathrm{Sym}(V)$, we have $\varphi(gx) = g\varphi(x) = gx$, so $gx \in \mathrm{Sym}(V)$. Similarly, if $y \in \mathrm{Alt}(V)$, we have $\varphi(gy) = g\varphi(y) = -gy$, so $gy \in \mathrm{Alt}(V)$. Thus, $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$ are submodules of $V \otimes V$.

To see that they are complementary subspaces, observe that if $x \in \mathrm{Sym}(V) \cap \mathrm{Alt}(V)$, then $\varphi(x) = x = -x$, and hence $x = 0$. We also observe that for every $v \in V$, we have

$$v = \frac{1}{2}(v + \varphi(v)) + \frac{1}{2}(v - \varphi(v))$$

and $\frac{1}{2}(v + \varphi(v)) \in \mathrm{Sym}(V)$ and $\frac{1}{2}(v - \varphi(v)) \in \mathrm{Alt}(V)$ (since φ^2 is the identity map). Therefore, $V \otimes V = \mathrm{Sym}(V) + \mathrm{Alt}(V)$.

(c) Find the characters χ_S and χ_A of $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$ in terms of the character χ of V, and verify that $\chi^2 = \chi_S + \chi_A$.

Solution: Given a basis $\{v_1, \ldots, v_n\}$ of V, a basis for $\mathrm{Sym}(V)$ is given by $\{v_i \otimes v_j + v_j \otimes v_i \mid i \leq j\}$ and a basis for $\mathrm{Alt}(V)$ is given by $\{v_i \otimes v_j - v_j \otimes v_i \mid i < j\}$.

Now, given any $g \in G$, let $\{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of G. Then if $gv_i = \lambda_i$ (so $\chi(g) = \sum_i \lambda_i$), we have

$$g(v_i \otimes v_j + v_j \otimes v_i) = \lambda_i v_i \otimes \lambda_j v_j + \lambda_j v_j \otimes \lambda_i v_i = \lambda_i \lambda_j (v_i \otimes v_j + v_j \otimes v_i)$$

and

$$g(v_i \otimes v_j - v_j \otimes v_i) = \lambda_i v_i \otimes \lambda_j v_j - \lambda_j v_j \otimes \lambda_i v_i = \lambda_i \lambda_j (v_i \otimes v_j - v_j \otimes v_i).$$

Therefore,

$$\chi_S(g) = \sum_{i \le j} \lambda_i \lambda_j, \qquad \chi_A(g) = \sum_{i<j} \lambda_i \lambda_j.$$

Since $g^2 v_i = \lambda_i^2 v_i$, this implies

$$\chi^2(g) = \left(\sum_i \lambda_i \right)^2 = \sum_i \lambda_i^2 + 2 \sum_{i<j} \lambda_i \lambda_j = \chi(g^2) + 2\chi_A(g),$$

and so $\chi_A(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$. Similarly, we can compute

$$\chi^2(g) = \left(\sum_i \lambda_i \right)^2 = 2 \sum_{i \le j} \lambda_i \lambda_j - \sum_i \lambda_i^2 = 2\chi_S(g) - \chi(g^2),$$

and so $\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$.

Note that by the previous part, we already expect $\chi^2 = \chi_S + \chi_A$. We can verify this directly now:

$$\chi_S(g) + \chi_A(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2)) + \frac{1}{2}(\chi^2(g) - \chi(g^2)) = \chi^2(g).$$

(d) Consider the character $\chi = \chi_4$ of S_4 given in the character table we constructed in lecture. Find a decomposition of χ^2 as a sum of irreducible characters. [This give us a way of decomposing the corresponding tensor product module as a direct sum of irreducible modules.]

Solution: We observe that $\chi^2(e) = |S_4|^2 = 24^2$ and $\chi^2(g) = 0$ for all $g \neq e$ in S_4. We now take the inner product of χ^2 with each of the characters in our table to find the multiplicity with which they appear in the decomposition of χ^2. Using the numbering from our character table for S_4 from lecture, we have

$$\begin{aligned}
\langle \chi^2, \chi_1 \rangle &= \frac{1}{24}(24^2) = 24 \\
\langle \chi^2, \chi_2 \rangle &= \frac{1}{24}(24^2) = 24 \\
\langle \chi^2, \chi_3 \rangle &= \frac{1}{24}(24^2)(2) = 48 \\
\langle \chi^2, \chi_4 \rangle &= \frac{1}{24}(24^2)(3) = 72 \\
\langle \chi^2, \chi_5 \rangle &= \frac{1}{24}(24^2)(3) = 72
\end{aligned}$$

and so $\chi^2 = 24\chi_1 + 24\chi_2 + 48\chi_3 + 72\chi_4 + 72\chi_5$.

3. (*From lecture*) Let N be a normal subgroup of G and let $\tilde{\chi}$ be a character of G/N. Let $\chi : G \to \mathbb{C}$ be given by $\chi(g) = \tilde{\chi}(gN)$. Show that χ is a character of G, and χ and $\tilde{\chi}$ have the same degree.

   Solution: Let $\tilde{\rho} : G/N \to \mathrm{GL}(V)$ be the representation associated to $\tilde{\chi}$, and define $\rho : G \to GL(V)$ by $\rho(g) = \tilde{\rho}(gN)$. We verify that ρ is a homomorphism. We observe

$$\rho(g)\rho(h) = \tilde{\rho}(gN)\tilde{\rho}(hN) = \tilde{\rho}(ghN) = \rho(gh).$$

   Thus, ρ is a homomorphism, and therefore a representation.

   Since $\rho(g) = \tilde{\rho}(gN)$, we also have that the trace $\chi(g)$ of $\rho(g)$ equals the trace of $\tilde{\chi}(gN)$ of $\tilde{\rho}(gN)$. Therefore, χ is a character of G satisfying $\chi(g) = \tilde{\chi}(gN)$.

   Since the associated module for both χ and $\tilde{\chi}$ are the same, these characters have the same degree.

4. Let G' denote the *commutator* subgroup of G, i.e. $G' = \langle xyx^{-1}y^{-1} \mid x, y \in G\rangle$. A standard fact in group theory is that the quotient group G/N is abelian if and only if $G' \subseteq N$. Show that the linear characters of G are precisely the lifts to G of the irreducible characters of G/G'. [This implies that there are exactly $|G/G'|$ linear characters of G.]

   Solution: Since G/G' is an abelian group, we know that all the characters of G/G' are of degree 1, hence linear. Therefore, by the previous part, we know that the lifts of characters of G/G' are also linear characters of G.

   Conversely, suppose we have a linear character λ of G. Then $\lambda : G \to \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^\times$ is a homomorphism, and by the first isomorphism theorem for groups, we have $G/\ker(\lambda) = \mathrm{im}(\lambda)$. Since $\mathrm{im}(\lambda)$ is a subgroup of $\mathbb{C}^\times$, it must be abelian, and so $G/\ker(\lambda)$ is abelian, and hence $G' subseteq \ker(\lambda)$. Therefore, we can define $\tilde{\lambda} : G/G' \to \mathrm{GL}(1, \mathbb{C})$ by $\tilde{\lambda}(gG') = \lambda(g)$. To see that this is well-defined, observe that if $gG' = hG'$, then $g^{-1}h \in G'$ and so $g^{-1}h \in \ker(\lambda)$ and so $\lambda(g^{-1}h) = 1$, which means $\lambda(g) = \lambda(h)$. Thus, $\tilde{\lambda}(gG') = \lambda(g) = \lambda(h) = \tilde{\lambda}(hG')$, and hence $\tilde{\lambda}$ is a well-defined homomorphism of G/G', and therefore one of its linear characters.

5. Find the character tables for

   (a) $D_4 = \langle r, f \mid r^4 = f^2 = e, fr = r^{-1}f\rangle$
   Solution:

$g_i:$	e	r	r^2	f	rf
$\lvert\text{Cl}(g_i)\rvert$	1	2	1	2	2
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

(b) $G = \langle a, b \mid a^6 = b^3 = 1, ba = ab^{-1} \rangle$.

Solution:

The 9 conjugacy classes of G are, for $0 \leq r \leq 2$, of the form:

$$\{a^{2r}\}, \{a^{2r}b, a^{2r}b^2\}, \{a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2\}.$$

We have $G' = \langle b \rangle$ and $G/G' = \langle aG' \rangle \cong C_6$. Hence we get 6 linear characters χ_j for $0 \leq j \leq 5$, as shown. The remaining characters must have degree 2, and their values can be computed using the orthogonality relations.

In the table below, we will compactify information by grouping together the conjugacy classes and characters as described above. Let $\omega = e^{2\pi i/6}$.

$g_i:$	a^{2r}	$a^{2r}b$	a^{2r+1}
$\lvert Z(g_i)\rvert$	18	9	6
χ_j $(0 \leq j \leq 5)$	ω^{2jr}	ω^{2jr}	$\omega^{j(2r+1)}$
ψ_k $(0 \leq k \leq 2)$	$2\omega^{2kr}$	$-\omega^{2kr}$	0

6. There exists a group G of order 10 with precisely four conjugacy classes with representatives g_1, g_2, g_3, g_4, and has an irreducible character χ given by

$g_i:$	g_1	g_2	g_3	g_4
χ	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0

(a) Find the sizes of the conjugacy classes of G. (*Hint:* It would be helpful to also have one other irreducible character for this.)

Solution: Let c_i be the size of the conjugacy class of g_i. Since only $\chi(g_1)$ is a positive integer, this must be the degree of the representation, and therefore $g_1 = e$. The identity element is in its own conjugacy class, and therefore $c_1 = 1$. Let τ be the trivial character. Then we have $\langle \tau, \tau \rangle = 1, \langle \chi, \chi \rangle = 1, \langle \chi, \tau \rangle = 0$, and therefore,

$$1 = \frac{1}{10}(1 + c_2 + c_3 + c_4)$$
$$1 = \frac{1}{10}\left(4 + c_2\frac{3-\sqrt{5}}{2} + c_3\frac{3+\sqrt{5}}{2}\right)$$
$$0 = \frac{1}{10}\left(2 + c_2\frac{-1+\sqrt{5}}{2} + c_3\frac{-1-\sqrt{5}}{2} + c_4\right).$$

Solving these equations, we get $c_2 = c_3 = 2$ and $c_4 = 5$.

(b) Complete the character table of G.

Solution: We can complete the first column using the sum of squares formula for the degrees of irreducible characters. Then, we observe that since g_4 is the only element with a conjugacy class of size 5, and its centraliser has order 2, it must be an element of order 2. Using orthogonality and Week 3 Q3a, we can complete the last column. The remaining entries can then be completed using orthogonality relations.

g_i :	g_1	g_2	g_3	g_4
$Z(g_i)$:	10	5	5	2
τ	1	1	1	1
χ	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0
χ_3	1	1	1	-1
χ_4	2	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$	0