LTCC: Representation Theory of Finite Groups
Exercise Set 4

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Suppose χ is a character of G and λ is a linear character of G.

   (a) Show that the product $\lambda\chi$ (given by $\lambda\chi(g) = \lambda(g)\chi(g)$) is also a character of G.

   (b) Show that if χ is irreducible, then so is $\lambda\chi$.

2. Let V and W be vector spaces. If $\{v_1, \dots, v_n\}$ is a basis for V and $\{w_1, \dots, w_m\}$ is a basis for W, then the *tensor product* $V \otimes W$ is the vector space with basis $\{v_i \otimes w_j \mid 1 \le i \le n, 1 \le j \le m\}$. [Note that for $v \in V$ and $w \in W$, we have $v \otimes w = (\sum_i \lambda_i v_i) \otimes (\sum_j \lambda_i v_i) = \sum_{i,j} \lambda_i \mu_j (v_i \otimes w_j)$.] If V and W are in fact $\mathbb{C}[G]$-modules, we can define an action of G on $V \otimes W$ by $g \cdot (v \otimes w) = gv \otimes gw$ and extending linearly.

   (a) Show that if the characters of V and W and χ and ψ, respectively, then the character of $V \otimes W$ is $\chi\psi$. [This shows that the product of any two characters of G is again a character of G. Note that this gives us an alternative proof of Exercise 1a, but Exercise 1a can also be solved more directly.]

   (b) Let V be a $\mathbb{C}[G]$-module with basis $\{v_1, \dots, v_n\}$, and let $\varphi : V \otimes V \to V \otimes V$ be the map given by $\varphi(v_i \otimes v_j) = v_j \otimes v_i$. Show that $\mathrm{Sym}(V) = \{x \in V \otimes V \mid \varphi(x) = x\}$ and $\mathrm{Alt}(V) = \{x \in V \otimes V \mid \varphi(x) = -x\}$ are complementary submodules of $V \otimes V$.

   (c) Find the characters χ_S and χ_A of $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$ in terms of the character χ of V, and verify that $\chi^2 = \chi_S + \chi_A$.

   (d) Consider the character $\chi = \chi_4$ of S_4 given in the character table we constructed in lecture. Find a decomposition of χ^2 as a sum of irreducible characters. [This give us a way of decomposing the corresponding tensor product module as a direct sum of irreducible modules.]

3. (*From lecture*) Let N be a normal subgroup of G and let $\tilde{\chi}$ be a character of G/N. Let $\chi : G \to \mathbb{C}$ be given by $\chi(g) = \tilde{\chi}(gN)$. Then χ is a character of G, and χ and $\tilde{\chi}$ have the same degree.

4. Let G' denote the *commutator* subgroup of G, i.e. $G' = \langle xyx^{-1}y^{-1} \mid x, y \in G \rangle$. A standard fact in group theory is that the quotient group G/N is abelian if and only if $G' \subseteq N$. Show that the linear characters of G are precisely the lifts to G of the irreducible characters of G/G'. [This implies that there are exactly $|G/G'|$ linear characters of G.]

5. Find the character tables for

   (a) $D_4 = \langle r, f \mid r^4 = f^2 = e, fr = r^{-1}f \rangle$

   (b) $G = \langle a, b \mid a^6 = b^3 = 1, ba = ab^{-1} \rangle$.

6. There exists a group G of order 10 with precisely four conjugacy classes with representatives g_1, g_2, g_3, g_4, and has an irreducible character χ given by

g_i :	g_1	g_2	g_3	g_4
χ	2	$\frac{-1+\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$	0

   (a) Find the sizes of the conjugacy classes of G. (*Hint:* It would be helpful to also have one other irreducible character for this.)

   (b) Complete the character table of G.