LTCC: Representation Theory of Finite Groups
Exercise Set 3

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Given a group G, show that character of the regular representation G is given by $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$ in G.

   Solution: Let ρ be the regular representation, and observe that for all $g \in G$, $\rho(g)$ permutes the elements of G, and hence is represented by a permutation matrix (with a single 1 in each row and column and 0s elsewhere). Therefore, the trace of $\rho(g)$ is given by the number of elements which are fixed under left multiplication by g. We observe that if $gh = h$ for some $h \in G$, then $g = e$. Therefore, since e fixes every element in G, we have $\chi(e) = |G|$ and since $g \neq e$ cannot fix any element in G, we have $\chi(g) = 0$.

2. Let S_n act on an n-dimensional space V with basis $\{v_1, \ldots, v_n\}$ by $\sigma v_i = v_{\sigma(i)}$. ($V$ is called the *permutation module* of S_n.)

   (a) Let χ be the character of the permutation representation of S_7. Find $\chi(x)$ for $x = (1\ 2)$ and $x = (1\ 6)(2\ 3\ 5)$.
   Solution: $\chi((1\ 2)) = 5$
   $\chi((1\ 6)(2\ 3\ 5)) = 2$

   (b) Show that in general the character χ of a permutation representation is given by $\chi(\sigma) = |\text{fix}(\sigma)|$ where $\text{fix}(g) = \{i \mid \sigma(i) = i\}$.
   Solution: Let ρ be the permutation representation. As in Q 1, for each $\sigma \in S_n$, the matrix $\rho(\sigma)$ will be a permuation matrix, and so $\chi(\sigma)$ equals the number of basis vectors fixed by the action of σ. This is precisely equal to $|\text{fix}(\sigma)|$.

3. Let $\rho : G \to GL(V)$ be a representation, and let χ be the character of ρ. Prove the following statements:

   (a) If $g \in G$ is an element of order 2, then $\chi(g)$ is an integer such that $\chi(g) \equiv \chi(e) \mod 2$.
   Solution: Since $g^2 = e$, we know that $\rho(g)^2 = I_V$. Therefore, the eigenvalues of g are all 1 or -1, and hence $\chi(g) = \text{tr}(\rho(g)) = k - \ell$ where k is the multiplicity of 1, ℓ is the multiplicity of -1 and $k + \ell = \dim(V) =: n$. Then, mod 2, we have $\chi(g) = k - \ell = k + \ell = n = \chi(e)$.

(b) $|\chi(g)| = \chi(e)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. (Note that the absolute value of a complex number is given by $|x + iy| = \sqrt{x^2 + y^2}$.)

Solution: If $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\chi(g) = \text{tr}(\rho(g)) = n\lambda$, and hence $|\chi(g)| = n|\lambda|$. But note that λ must be a root of unity since $\rho(g)$ has finite order, and so $|\lambda| = 1$, and hence $|\chi(g)| = n = \chi(e)$.

Now suppose $|\chi(g)| = n = \chi(e)$. Then $\chi(g)$, which is the sum of m^{th} roots of unity, must in fact be the sum of n copies of the same root of unity ω. We know there is a basis in which $\rho(g)$ is diagonal (for example, by the proof of Section 3.1, Prop 3), and so in this basis $\rho(g) = (diag)(\omega, \omega, \ldots, \omega) = \omega I_n$. But any change of basis (conjugation) would leave this matrix unchanged, and hence $\rho = \omega I_n$ in any basis.

(c) $\ker(\rho) = \{g \in G | \chi(g) = \chi(e)\}$

Solution: Suppose $g \in \ker(\rho)$. Then $\rho(g) = I_n = \rho(e)$, and hence $\chi(g) = \chi(e)$. Now suppose $\chi(g) = \chi(e)$ for some $g \in G$. Then by the previous part, we know that $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. Therefore $\chi(g) = n\lambda$, but this can only equal $\chi(e) = n$ if $\lambda = 1$. Thus, $\rho(g) = I_n$ and hence $g \in \ker(\rho)$.

4. Prove that if χ is a faithful irreducible character if G, then the center of G is given by $Z(G) = \{g \in G \mid |\chi(g)| = \chi(e)\}$.

   Solution: Let ρ be the corresponding representation.

   Suppose g satisfies $|\chi(g)| = \chi(e)$. If $g \in Z(G)$, then since ρ is irreducible, by Week 1 Q4a we have $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$. Conversely, if $g = \lambda I_n$ for some $\lambda \in \mathbb{C}$, then $\rho(g)$ commutes with $\rho(h)$ for all $h \in G$, and hence $g \in Z(G)$ since ρ is faithful.

   Finally, by Q2b, we know that $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$ iff $|\chi(g)| = \chi(e)$, and so the result follows.

5. Let χ be an irreducible character of G, and suppose $z \in Z(G)$ is an element of order m. Show that there exists an m^{th} root of unity $\lambda \in \mathbb{C}$ such that for all $g \in G$, $\chi(zg) = \lambda\chi(g)$.

   Solution: Let ρ be the corresponding representation. By Q2b, we know that $\rho(z) = \lambda I_n$ for some $\lambda \in \mathbb{C}$, and since $\rho(z^m) = \rho(z)^m = I_n$, we know λ must be an m^{th} root of unity. Therefore $\rho(zg) = \rho(z)\rho(g) = \lambda\rho(g)$ for all $g \in G$, and hence $\chi(zg) = \lambda\chi(g)$.

6. Let χ be a character of G. Show that χ is a homorphism from G to $\mathbb{C}^\times$ if and only if χ is the character of a degree one representation. (Such characters are called *linear characters.*)

   Solution: If χ has degree $n > 1$, then $\chi(e) = n$, so χ does not take identity in G to identity in $\mathbb{C}^\times$, and hence cannot be a homomorphism.

Now suppose χ has degree 1, so that $\chi(e) = 1$. Then if ρ is the corresponding representation, we have $\chi(g) = \rho(g)$, and a representation of G is a group homomorphism by definition. Thus, χ is in fact a homomorphism in this case.

7. Suppose χ is a nonzero, nontrivial character of G, and that $\chi(g)$ is a nonnegative real number for all $g \in G$. Show that χ must be reducible.

   Solution: Let τ be the trivial character of G. Then we have

$$\langle \chi, \tau \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g).$$

   Since $\chi(g)$ is a nonnegative real number for all $g \in G$ and χ is nonzero, we get that $\langle \chi, \tau \rangle > 0$, so τ is a constituent of χ. Since χ is given to be nontrivial itself, it must therefore be reducible.