LTCC: Representation Theory of Finite Groups
Exercise Set 3

Throughout this exercise set, assume G is a finite group, and that we are working over the field of complex numbers.

1. (*From lecture*) Given a group G, show that character of the regular representation G is given by $\chi(e) = |G|$ and $\chi(g) = 0$ for all $g \neq e$ in G.

2. Let S_n act on an n-dimensional space V with basis $\{v_1, \ldots, v_n\}$ by $\sigma v_i = v_{\sigma(i)}$. ($V$ is called the *permutation module* of S_n.)

   (a) Let χ be the character of the permutation representation of S_7. Find $\chi(x)$ for $x = (1\ 2)$ and $x = (1\ 6)(2\ 3\ 5)$.

   (b) Show that in general the character χ of a permutation representation is given by $\chi(\sigma) = |\text{fix}(\sigma)|$ where $\text{fix}(g) = \{i \mid \sigma(i) = i\}$.

3. Let $\rho : G \to GL(V)$ be a representation, and let χ be the character of ρ. Prove the following statements:

   (a) If $g \in G$ is an element of order 2, then $\chi(g)$ is an integer such that $\chi(g) \equiv \chi(e) \mod 2$.

   (b) $|\chi(g)| = \chi(e)$ if and only if $\rho(g) = \lambda I_n$ for some $\lambda \in \mathbb{C}$. (Note that the absolute value of a complex number is given by $|x + iy| = \sqrt{x^2 + y^2}$.)

   (c) $\ker(\rho) = \{g \in G | \chi(g) = \chi(e)\}$

4. Prove that if χ is a faithful irreducible character if G, then the center of G is given by $Z(G) = \{g \in G \mid |\chi(g)| = \chi(e)\}$.

5. Let χ be an irreducible character of G, and suppose $z \in Z(G)$ is an element of order m. Show that there exists an m^{th} root of unity $\lambda \in \mathbb{C}$ such that for all $g \in G$, $\chi(zg) = \lambda\chi(g)$.

6. Let χ be a character of G. Show that χ is a homorphism from G to $\mathbb{C}^\times$ if and only if χ is the character of a degree one representation. (Such characters are called *linear characters.*)

7. Suppose χ is a nonzero, nontrivial character of G, and that $\chi(g)$ is a nonnegative real number for all $g \in G$. Show that χ must be reducible.