LTCC: Representation Theory of Finite Groups
Exercise Set 2

1. (*From lecture:*) Let A be the group algebra over $\mathbb{C}$ of $G = C_3 = \langle x | x^3 = e \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be an A-module with representation $\rho(v_i) = v_{i+1 \mod 3}$. This module has a submodule $U = \langle v_1 + v_2 + v_3 \rangle$. Find a submodule W such that $V = U \oplus W$.

   Solution: Let $W_0 = \langle v_1, v_2 \rangle$, so that $V = U \oplus W_0$ as vector spaces (note that W_0 is not a submodule). Let $\varphi : V \to U$ be the projection map from $V = U \oplus W_0$ to U. We can verify that for all basis vectors v_i, we have

$$\psi(v_i) = \frac{1}{|G|} \sum_{g \in G} g \varphi g^{-1}(v_i) = \frac{1}{3}(v_1 + v_2 + v_3).$$

   Thus, $W = \ker(\psi) = \left\{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \right\}$.

2. Is the submodule W from the previous exercise irreducible?

   Solution: The submodule $W = \left\{ \sum \lambda_i v_i \mid \sum \lambda_i = 0 \right\}$ is two dimensional, with a possible basis given by $v_1 - v_2, v_2 - v_3$. We have seen (as a corollary to Schur's Lemma) that all the irreducible submodules of an abelian group must be 1-dimensional, so this implies that W is not irreducible.

3. Let V, W_1, W_2 be A-modules. Show that

$$\mathrm{Hom}_A(V, W_1 \oplus W_2) \cong \mathrm{Hom}_A(V, W_1) \oplus \mathrm{Hom}_A(V, W_2)$$

   as vector spaces (and thus their dimensions are the same).

   Solution: Let $W = W_1 \oplus W_2$ and let π_1, π_2 be the projection maps from W to W_1, W_2, respectively, and note that the π are A-homomorphisms. Define a map $\varphi : \mathrm{Hom}_A(V, W_1 \oplus W_2) \to \mathrm{Hom}_A(V, W_1) \oplus \mathrm{Hom}_A(V, W_2)$ by $\varphi(f) = (\pi_1 \circ f, \pi_2 \circ f)$. We verify that φ is linear and bijective, and therefore an isomorphism of vector spaces.

4. Let G be a group. Suppose that ρ is a representation of G on a vector space V of dimension 1. Prove that $G/\ker(\rho)$ is abelian.

   Solution: We note that $G/\ker(\rho) \cong \mathrm{im}(\rho)$ (as groups), and since $\mathrm{im}(\rho) \subset GL(V) \cong \mathbb{F}^\times$ (since V is 1-dimensional), we see that $\mathrm{im}(\rho)$ is a subgroup of an abelian group, and hence also abelian.

5. Let $\phi : G \to GL(n, \mathbb{F})$ be a representation of G. Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

is also a representation of G.

Solution: We observe that

$$\rho(gh) = \det(\phi(gh)) = \det(\phi(g)\phi(h)) = \det(\phi(g))\det(\phi(h)) = \rho(g)\rho(h),$$

and

$$\rho(e) = \det(\phi(e)) = \det(I_n) = 1.$$

Therefore, ρ is also a representation.

6. Let $\rho : G \to GL(n, \mathbb{F})$ be a representation of G. Show that $\sigma : G/\ker(\rho) \to GL(n, \mathbb{F})$ given by $\sigma(g\ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Recall from Exercise Sheet 1 that a faithful representation is injective. Note that here you must show that it is a well-defined representation, as well as showing that it is faithful.)

Solution: Suppose $g\ker(\rho) = h\ker(\rho)$. This equivalent to the condition that $g^{-1}h \in \ker(\rho)$ and hence $\rho(g^{-1}h) = I_n$. Since ρ is a representation, we have $\rho(g)^{-1}\rho(h) = I_n$, and hence $\rho(g) = \rho(h)$. Thus, $\sigma(g\ker(\rho)) = \sigma(h\ker(\rho))$, so σ is well-defined.

We also observe that

$$\sigma(g\ker(\rho)h\ker(\rho)) = \sigma(gh\ker(\rho)) = \rho(gh) = \rho(g)\rho(h) = \sigma(g\ker(\rho))\sigma(h\ker(\rho)),$$

and

$$\sigma(e\ker(\rho)) = \rho(e) = I_n.$$

Therefore, σ is also a representation.

Finally, we suppose $\sigma(g\ker(\rho)) = \sigma(h\ker(\rho))$. Then $\rho(g) = \rho(h)$, and so $\rho(g)^{-1}\rho(h) = I_n$, and therefore $\rho(g^{-1}h) = I_n$. Thus, $g^{-1}h \in \ker(\rho)$, and so $g\ker(\rho) = h\ker(\rho)$. Therefore, σ is a faithful representation.

7. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}[G]$-module.

Solution: By Maschke's Theorem, the regular $\mathbb{C}[G]$-module decomposes into irreducible submodules. Let ρ be the representation of G on one of the nontrivial submodules. (Note that nontrivial submodules must exist whenever G is a nontrivial group. When G is trivial, the trivial module is faithful.) We then have that action of G on this nontrivial irreducible module is faithful, because $\ker(\rho)$ is normal in G, and therefore must be trivial.

8. When constucting the regular representation of G over $\mathbb{F}$, we used the action of G on itself by *left multiplication*, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.

   (a) Define the *right multiplication* action $\cdot : G \times G \to G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.

   Solution: We observe:

$$gh \cdot x = x(gh)^{-1} = xh^{-1}g^{-1} = g \cdot (h \cdot x)$$

   and $e \cdot x = xe^{-1} = x$. Thus, this map is an action of G on $\mathbb{F}[G]$. This fails for the operation $g \cdot x = xg$, since in this case:

$$gh \cdot x = x(gh) = h \cdot (g \cdot x).$$

   (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.

   Solution: Let $T : \mathbb{F}[G] \to \mathbb{F}[G]$ be the map given by $x \mapsto x^{-1}$ for all $x \in G$. (This is a permutation of basis vectors of $\mathbb{F}[G]$ and hence an invertible linear map.) We observe that for all $g \in G$, $g \cdot T(x) = g \cdot x^{-1} = x^{-1}g^{-1} = (gx)^{-1} = T(gx)$. Thus, T is an equivalence of representations.

   (c) Define the *conjugation* action $\cdot : G \times G \to G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.

   Solution: We observe:

$$gh \cdot x = (gh)x(gh)^{-1} = ghxh^{-1}g^{-1} = g \cdot (h \cdot x)$$

   and $e \cdot x = exe^{-1} = x$. Thus, this map is an action of G on $\mathbb{F}[G]$.

   (d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)

   Solution: Let $G = C_3$. We observe that every element of G is fixed by the conjugation action of G, and hence so is every element of $\mathbb{F}[G]$, so $\mathbb{F}[G]$ decomposes into the direct sum of three copies of the trivial module. On the other hand, the G acts nontrivially on the regular $\mathbb{F}[G]$-module.

   Alternative solution: Let $|G| > 1$. Suppose there exists a bijective linear map $T : \mathbb{F}[G] \to \mathbb{F}[G]$ such that $T(gx) = gT(x)g^{-1}$ for all $g \in G, x \in \mathbb{F}[G]$. By setting $x = e$, we get $T(g) = gT(e)g^{-1} = e$ for all $g \in G$, which contradicts the fact that G is bijective.

9. Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$-module has dimension 1. Show that G is abelian.

   Solution: By Maschke's Theorem, the regular $\mathbb{C}[G]$-module decomposes into irreducible submodules. Thus, we get that $\mathbb{C}[G] = \bigoplus_{i=1}^{n} U_i$ where each U_i has dimension 1 in this case. For each i let u_i be a nonzero vector in U_i. Then $\{u_1, \ldots, u_n\}$ is a basis for $\mathbb{C}[G]$, and for each $g \in G, i \in I$, we have $gu_i = \lambda_i u_i$ for some $\lambda_i \in \mathbb{C}$. Thus, g is a diagonal matrix in this basis for all $g \in G$. Since any two diagonal matrices commute, we get that $gh = hg$ for all $g, h \in G$, and so the group is abelian.

10. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.

    Solution: Suppose ρ is reducible. Then there exists a proper nonzero $\mathbb{C}[G]$-submodule V of $\mathbb{C}^2$, which means that V is 1-dimensional. By Maschke's Theorem, there also exists a submodule W such that $\mathbb{C}^2 = V \oplus W$. Let v, w be nonzero vectors of V and W, respectively. Note that $\{v, w\}$ is a basis of $\mathbb{C}^2$ in this case, and so $\rho(g)$ and $\rho(h)$ act diagonally with respect to this basis and hence commute, which is a contradiction. Thus, ρ must be irreducible.