LTCC: Representation Theory of Finite Groups
Exercise Set 2

1. (*From lecture:*) Let A be the group algebra over $\mathbb{C}$ of $G = C_3 = \langle x | x^3 = e \rangle$ and let $V = \langle v_1, v_2, v_3 \rangle$ be an A-module with representation $\rho(v_i) = v_{i+1 \mod 3}$. This module has a submodule $U = \langle v_1 + v_2 + v_3 \rangle$. Find a submodule W such that $V = U \oplus W$.

2. Is the submodule W from the previous exercise irreducible?

3. Let V, W_1, W_2 be A-modules. Show that

$$\mathrm{Hom}_A(V, W_1 \oplus W_2) \cong \mathrm{Hom}_A(V, W_1) + \mathrm{Hom}_A(V, W_2)$$

   as vector spaces (and thus their dimensions are the same).

4. Let G be a group. Suppose that ρ is a representation of G on a vector space V of dimension 1. Prove that $G/\ker(\rho)$ is abelian.

5. Let $\phi : G \to GL(n, \mathbb{F})$ be a representation of G. Prove that the map

$$\rho : g \mapsto \det(\phi(g))$$

   is also a representation of G.

6. Let $\rho : G \to GL(n, \mathbb{F})$ be a representation of G. Show that $\sigma : G/\ker(\rho) \to GL(n, \mathbb{F})$ given by $\sigma(g \ker(\rho)) = \rho(g)$ is a faithful representation of $G/\ker(\rho)$. (Recall from Exercise Sheet 1 that a faithful representation is injective. Note that here you must show that it is a well-defined representation, as well as showing that it is faithful.)

7. A *simple* group is a group that has no nontrivial, proper normal subgroups. Prove that for every finite simple group G, there exists a faithful irreducible $\mathbb{C}[G]$-module.

8. When constucting the regular representation of G over $\mathbb{F}$, we used the action of G on itself by *left multiplication*, ie $g \cdot x = gx$ for $g, x \in G$. In this problem, we will use other actions of G on itself to construct new representations with associated module $\mathbb{F}[G]$.

   (a) Define the *right multiplication* action $\cdot : G \times G \to G$ by $g \cdot x = xg^{-1}$. Show that right multiplication is in fact an action of G on itself. Explain why the operation $g \cdot x = xg$ is not an action.

   (b) Show that the representation given by right multiplication is equivalent to the representation given by left multiplication.

(c) Define the *conjugation* action $\cdot : G \times G \to G$ by $g \cdot x = gxg^{-1}$. Show that conjugation is in fact an action of G on itself.

(d) Show that in general, conjugation is not equivalent to left multiplication. (Hint: Consider $G = C_3$.)

9. Suppose G is a finite group such that every irreducible $\mathbb{C}[G]$-module has dimension 1. Show that G is abelian.

10. Let G be a finite group and let $\rho : G \to GL(2, \mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Prove that ρ is irreducible.