LTCC: Representation Theory of Finite Groups
Exercise Set 1

1. (*From lecture:*) Let A be an associative algebra and let V_1, V_2 be A-modules. Let $T : V_1 \to V_2$ be an intertwining operator. Show that $\ker(T)$ is a submodule of V_1 and that $\operatorname{im}(T)$ is a submodule of V_2.

   Solution: Since T is a linear map, we already know that $\ker(T)$ is a subspace of V_1 and $\operatorname{im}(T)$ is a subspace of V_2.

   Then we observe that if $v \in \ker(T)$, then for all $a \in A$, $T(av) = aT(v) = a0 = 0$, and hence $av \in \ker(T)$. Thus, $\ker(T)$ is a submodule of V_1.

   Then we observe that if $w \in \operatorname{im}(T)$, then $w = T(v)$ for some $v \in V_1$, and hence for any $a \in A$, $aw = aT(v) = T(av)$, and hence $aw \in \operatorname{im}(T)$ also. Thus, $\operatorname{im}(T)$ is a submodule of V_2.

2. Let $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$. We define the following representations of $\mathbb{R}[D_4]$ on $\mathbb{R}^2$:

$$\rho((f^j r^k) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k$$
$$\sigma((f^j r^k) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k$$

   Show that ρ and σ are isomorphic representations. (Please specify an explicit linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that shows this equivalence.)

   Solution: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. T is clearly a bijective linear map. It thus suffices to check that T commutes with the action of r and f, since these are the multiplicative generators of the algebra. We observe that

$$T\left(\rho(r)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} -y \\ x \end{bmatrix}\right) = \begin{bmatrix} x \\ -y \end{bmatrix}$$

   and
$$\sigma(r)\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

   We also have

$$T\left(\rho(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} -x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$$

and

$$\sigma(f)\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}.$$

Thus, T is an isomorphism of representations.

3. Let V be a nonzero finite dimensional A-module. Show that it has an irreducible submodule. Then show by example that this does not always hold for infinite dimensional representations.

   Solution: For finite dimensional modules, we use induction on the dimension of the module. If V is 1-dimensional, then it is automatically irreducible (hence an irreducible submodule of itself). Now suppose the result holds for modules with dimension less than some fixed n, and let V be an n-dimensional A-module. Then either V is irreducible itself, or it has a nonzero proper submodule, U. Since $\dim(U) < n$, by the inductive hypothesis, U has an irreducible submodule, which is also an irreducible submodule of V.

   Now let $A = \mathbb{F}[x]$, and let $V = A$ be the regular A-module. We claim that V has no irreducible submodules. To see this, suppose U is a nonzero submodule of V. Then U contains a nonzero vector, say, v, and therefore contains all of Av. But since v is nonzero, then xv is nonzero, so Axv is a nonzero submodule of Av. But $v \notin Axv$ (compare lowest degree terms with nonzero coefficient), and so Axv is a nonzero proper submodule of Av, and therefore Av, and hence also U, is not irreducible.

4. Let A be an algebra over an algebraically closed field $\mathbb{F}$. The center $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A. Note that if A is commutative, then $Z(A) = A$.

   (a) Show that if V is an irreducible finite dimensional A-module, then any element $z \in Z(A)$ acts on V by multiplication by some scalar $\chi_V(z)$. Show that $\chi_V : Z(A) \to \mathbb{F}$ is a homomorphism. (This homomorphism is called the **central character** of V.)

   Solution: Since every $z \in Z(A)$ commutes with the action of A, i.e. $(az)v = (za)v$ for all $a \in A$, we have that the map $v \mapsto zv$ is an intertwining operator. Thus, by Schur's lemma for algebraically closed fields, z acts by a scalar λ_z on V.

   The map $\chi_V : Z(A) \to \mathbb{F}$ given by $z \mapsto \lambda_z$ is also immediately a homomorphism since it is a linear map that commutes with the action of A.

   (b) Show that if V is an indecomposable finite dimensional A-module, then for any $z \in Z(A)$, the operator $\rho(z)$ by which z acts on V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible submodule of

V. Thus $\chi_V : Z(A) \to \mathbb{F}$ is a homomorphism, which is again called the central character of V.

Solution: Fix $z \in Z(A)$. Then V has a basis of generalized eigenvectors and is a direct sum of generalized eigenspaces of z. Let λ be an eigenvalue of z and suppose $(z - \lambda 1)^m v = 0$. Then $(z - \lambda 1)^m av = a(z - \lambda 1)^m v = 0$, for all $a \in A$, since $(z - \lambda 1)^m$ is also in the centre. Therefore, each generalized eigenspace is a submodule of V. Since V is indecomposable, there can only be one of these, and hence there is a single eigenvalue of z. By the previous exercise, V has an irreducible submodule U, and by part (a), z acts on U by a scalar, which must then equal this eigenvalue.

(c) Does $\rho(z)$ have to be a scalar operator?

Solution: Even if V is indecomposable, $\rho(z)$ need not act by a scalar, as we have seen by example in Section 1.2, when $A = \mathbb{F}[x]$ and $V = \mathbb{F}^2$, with $\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

5. Let A be an associative algebra, and let V be an A-module. By $\mathrm{End}_A(V)$ we denote the algebra of all homomorphisms of representations $V \to V$. Show that $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$, the algebra A with opposite multiplication. [Here, we take A to act on itself via the regular representation.]

   Solution: Define a map $T : \mathrm{End}_A(A) \to A^{\mathrm{op}}$ by $T(\phi) = \phi(1)$ (i.e. the map evaluates elements in $\mathrm{End}_A(A)$ at 1). An evaluation map is linear. We can verify that for $\phi, \psi \in \mathrm{End}_A(A)$, if $\phi(1) = a, \psi(1) = b$, we have $T(\phi \circ \psi) = \phi(\psi(1)) = \phi(b) = \phi(b1) = b\phi(1) = ba = T(\psi)T(\phi)$.

6. Let $C_n = \langle x \rangle$ be the cyclic group of order n generated by x. For $0 \leq j < n$, let $\rho_j : \mathbb{C}[C_n] \to End(\mathbb{C}) \cong \mathbb{C}$ be the map given by

$$\rho_j(x^t) = e^{2\pi ijt/n}.$$

   (Note that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.)

   (a) For which values of j is ρ_j a representation of $\mathbb{C}[C_n]$?

   Solution: Since x^n is the identity element in C_n, we require $\rho_j(x^n) = e^{2\pi ij} = 1$. This is satisfied by all j, and thus ρ_j a representation of $\mathbb{C}[C_n]$ for all $0 \leq j < n$.

   (b) We say a representation is **faithful** if it is injective. For which values of j is ρ_j a faithful representation of $\mathbb{C}[C_n]$?

   Solution: For ρ_j to be faithful, we require $\rho_j(x^t) \neq 1$ for $0 < t < n$, which means $e^{2\pi ijt/n} \neq 1$ for $0 < t < n$. This is satisfied by all j coprime to n.

7. Suppose V is an A-module and W is a submodule of V. Show that V/W is also an A-module.

   Solution: Define an action of A on V/W by $a(v+W) = av+W$. To check that this is well-defined, suppose $v+W = v'+W$. Then $v-v' \in W$. Since W is a submodule, we have $a(v-v') \in W$, and hence $av-av' \in W$. Thus $av+W = av'+W$.