LTCC: Representation Theory of Finite Groups
Exercise Set 1

1. (*From lecture:*) Let A be an associative algebra and let V_1, V_2 be A-modules. Let $T : V_1 \to V_2$ be an intertwining operator. Show that $\ker(T)$ is a submodule of V_1 and that $\mathrm{im}(T)$ is a submodule of V_2.

2. Let $D_4 = \{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$. We define the following representations of $\mathbb{R}[D_4]$ on $\mathbb{R}^2$:

$$\rho((f^j r^k) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^k$$

$$\sigma((f^j r^k) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^j \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^k$$

Show that ρ and σ are isomorphic representations. (Please specify an explicit linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that shows this equivalence.)

3. Let V be a nonzero finite dimensional A-module. Show that it has an irreducible submodule. Then show by example that this does not always hold for infinite dimensional representations.

4. Let A be an algebra over an algebraically closed field $\mathbb{F}$. The center $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A. Note that if A is commutative, then $Z(A) = A$.

   (a) Show that if V is an irreducible finite dimensional A-module, then any element $z \in Z(A)$ acts on V by multiplication by some scalar $\chi_V(z)$. Show that $\chi_V : Z(A) \to \mathbb{F}$ is a homomorphism. (This homomorphism is called the **central character** of V.)

   (b) Show that if V is an indecomposable finite dimensional A-module, then for any $z \in Z(A)$, the operator $\rho(z)$ by which z acts on V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible submodule of V. Thus $\chi_V : Z(A) \to \mathbb{F}$ is a homomorphism, which is again called the central character of V.

   (c) Does $\rho(z)$ have to be a scalar operator?

5. Let A be an associative algebra, and let V be an A-module. By $\mathrm{End}_A(V)$ we denote the algebra of all homomorphisms of representations $V \to V$. Show that $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$, the algebra A with opposite multiplication. [Here, we take A to act on itself via the regular representation.]

6. Let $C_n = \langle x \rangle$ be the cyclic group of order n generated by x. For $0 \leq j < n$, let $\rho_j : \mathbb{C}[C_n] \to End(\mathbb{C}) \cong \mathbb{C}$ be the map given by

$$\rho_j(x^t) = e^{2\pi i j t/n}.$$

(Note that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.)

   (a) For which values of j is ρ_j a representation of $\mathbb{C}[C_n]$?

   (b) We say a representation is **faithful** if it is injective. For which values of j is ρ_j a faithful representation of $\mathbb{C}[C_n]$?

7. Suppose V is an A-module and W is a submodule of V. Show that V/W is also an A-module.