LTCC: Representation Theory of Finite Groups
Mock Exam

1. Let $G = C_3 = \langle a \rangle$, and define a map $\rho : G \to GL(2, \mathbb{C})$ by

$$\rho(a) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

   (a) Verify that ρ is a representation of G.

   We need to check that $\rho(a)^3 = \rho(a^3) = \rho(e) = I_2$. We observe

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Thus, ρ is a well-defined representation of G.

   (b) Decompose the corresponding module $\mathbb{C}^2$ into a direct sum of irreducible $\mathbb{C}[G]$-submodules.

   Any nonzero proper submodule of $\mathbb{C}^2$ has to have dimension 1, and hence is an eigenspace of $\rho(a)$. The characteristic equation of $\rho(a)$ is

$$(-1 - t)(-t) + 1 = t^2 + t + 1 = 0,$$

   so $t = e^{\pm 2\pi i/3}$. Let $\omega = e^{2\pi i/3}$. Then we wish to find x, y satisfying:

$$\rho(a) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x - y \\ x \end{bmatrix} = \omega^{\pm} \begin{bmatrix} x \\ y \end{bmatrix}.$$

   Thus, the eigenvectors corresponding to ω and ω^{-1} are $\begin{bmatrix} \omega \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \omega^{-1} \\ 1 \end{bmatrix}$, respectively, and therefore

$$\mathbb{C}^2 = \left\langle \begin{bmatrix} \omega \\ 1 \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} \omega^{-1} \\ 1 \end{bmatrix} \right\rangle.$$

2. Give examples, with brief justification, of each of the following:

   (a) a finite group with an irreducible representation of degree greater than 1 over $\mathbb{C}$

   We have seen in the character table of D_3 that it has an irreducible representation of degree 2.

(b) a finite group with no faithful irreducible representations over $\mathbb{C}$ $G = C_2 \times C_2$ has no faithful irreducible representation. To see this, note that the only irreducible characters of an abelian group are linear characters (in which the character equals the representation), and since every element has order 2, we have $\chi(g) = \pm 1$ for all $g \in C_2 \times C_2$. Thus, since $|\mathrm{im}(\chi)| = 2$ and $|G| = 4$, we must have $\ker(\chi) = 2$, and hence the representation is not injective.

3. Find the character table of the group $G = \langle a, b \mid a^6 = e, a^3 = b^2, bab^{-1} = a^{-1} \rangle$.

g_i :	e	a^3	a	a^2	b	ab
$\lvert Z(g_i) \rvert$	12	12	6	6	4	4
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	i	$-i$
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	-1	1	$-i$	i
χ_5	2	2	-1	-1	0	0
χ_6	2	-2	1	-1	0	0

4. Let G be a group that acts on $X = \{1, 2, \ldots, n\}$ by permutations, and let $V = \mathrm{span}\{v_1, \ldots, v_n\}$ be the corresponding permutation module of G. Let χ be the character corresponding to V and let τ be the trivial character of G. Then one can show (using a result known as Burnside's Lemma) that if c is the number of distinct orbits of the action of G on X, then $\langle \chi, \tau \rangle = c$ and $\chi = c\tau + \psi$ where $\langle \psi, \tau \rangle = 0$.

(a) Let G act on $X \times X$ by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Show that the corresponding permutation module has character χ^2.

The corresponding permutation module is given by the tensor product $V \otimes V$, which we have seen in HW 4 has character χ^2.

(b) Now suppose $G = S_n$. Show that the action of G on X has exactly one orbit and that the action of G on $X \times X$ has exactly two orbits.

Since the element $(1\ k) \in S_n$ takes 1 to k, we have that every $k \in X$ is in the orbit of 1. Thus, the action of G on X has exactly one orbit.

Since the element $(1\ k) \in S_n$ takes 1 to k, we have that every $(k, k) \in X \times X$ is in the orbit of $(1, 1)$. However, if $x \neq y$, we can show that any pair (a, b) with $a \neq b$ is in the orbit of (x, y) by acting by the element $(x\ a)(y\ b)$ (omiting the transposition as appropriate if $x = a$ or $y = b$). However, since $gx \neq gy$ for all permutations $g \in S_n$, we also see that these two orbits are distinct.

(c) Show that $\langle \chi^2, \tau \rangle = 2$ for the permutation character χ of S_n.

Since the number of distinct orbits of the action of G on $X \times X$ is 2, and the character corresponding to this action is χ^2, we have $\langle \chi^2, \tau \rangle = 2$.

(d) Show that the *standard* module of S_n (i.e. the complement of the trivial module inside the permutation module) is irreducible.

Let ψ be the character of the standard module, so that $\chi = \tau + \psi$. Then we observe that:

$$\begin{aligned} \langle \chi, \chi \rangle &= \langle \tau + \psi, \tau + \psi \rangle \\ &= \langle \tau, \tau \rangle + 2 \langle \tau, \psi \rangle + \langle \psi, \psi \rangle \\ &= 1 + \langle \psi, \psi \rangle . \end{aligned}$$

On the other hand, we also have $\langle \chi, \chi \rangle = \langle \chi^2, \tau \rangle$ since χ is a real character, and so

$$1 + \langle \psi, \psi \rangle = 2,$$

and hence $\langle \psi, \psi \rangle = 1$. Thus, ψ is irreducible.

5. (a) Let H be the trivial subgroup of G, and let ψ be the trivial character of H. Show that $\psi \uparrow G$ is the regular character of G.

We have

$$\begin{aligned} \psi \uparrow G(g) &= \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(y^{-1} g y) \\ &= \sum_{y \in G} \dot{1}(y^{-1} g y). \end{aligned}$$

But $\dot{1}(y^{-1}gy) = 1$ iff $y^{-1}gy = e$ (and 0 otherwise), and $y^{-1}gy = e$ iff $g = e$. Thus, $\psi \uparrow Gg) = |G|$ for $g = e$ and 0 if $g \neq e$.

Let H be any subgroup of G. Show that each irreducible representation of G is contained in a representation induced from an irreducible representation of H.

Let χ be an irreducible character of G, and let ψ be an irreducible character of H that is a constituent of $\chi \downarrow H$. Then by Frobenius Reciprocity, we have

$$\langle \chi \downarrow H, \psi \rangle = \langle \chi, \psi \uparrow G \rangle .$$

Since $\langle \chi \downarrow H, \psi \rangle \neq 0$, we have $\langle \chi, \psi \uparrow G \rangle \neq 0$, and thus, χ is a constituent of the character induced from the irreducible character ψ.

(b) Let H be an abelian subgroup of G with index n. Show that the degree of each irreducible character χ of G is at most n.

Let χ be an irreducible character of G. Then by the previous part, χ is a constituent of a character induced from an irreducible character ψ of H. Since H is abelian, note that ψ has degree 1, i.e. $\psi(e) = 1$. Therefore, we have

$$\begin{aligned} \psi \uparrow G(e) &= \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(y^{-1}y) \\ &= \frac{|G|}{|H|} \psi(e) = n. \end{aligned}$$

Since χ is a constituent of $\psi \uparrow G$, we have $\chi(e) \leq n$.