LTCC: Representation Theory of Finite Groups
Mock Exam

1. Let $G = C_3 = \langle a \rangle$, and define a map $\rho : G \to GL(2, \mathbb{C})$ by

$$\rho(a) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

   (a) Verify that ρ is a representation of G.

   (b) Decompose the corresponding module $\mathbb{C}^2$ into a direct sum of irreducible $\mathbb{C}[G]$-submodules.

2. Give examples, with brief justification, of each of the following:

   (a) a finite group with an irreducible representation of degree greater than 1 over $\mathbb{C}$

   (b) a finite group with no faithful irreducible representations over $\mathbb{C}$

3. Find the character table of the group $G = \langle a, b \mid a^6 = e, a^3 = b^2, bab^{-1} = a^{-1} \rangle$.

4. Let G be a group that acts on $X = \{1, 2, \ldots, n\}$ by permutations, and let $V = \operatorname{span}\{v_1, \ldots, v_n\}$ be the corresponding permutation module of G. Let χ be the character corresponding to V and let τ be the trivial character of G. Then one can show (using a result known as Burnside's Lemma) that if c is the number of distinct orbits of the action of G on X, then $\langle \chi, \tau \rangle = c$ and $\chi = c\tau + \psi$ where $\langle \psi, \tau \rangle = 0$.

   (a) Let G act on $X \times X$ by $g \cdot (x, y) = (g \cdot x, g \cdot y)$. Show that the corresponding permutation module has character χ^2.

   (b) Now suppose $G = S_n$. Show that the action of G on X has exactly one orbit and that the action of G on $X \times X$ has exactly two orbits.

   (c) Show that $\langle \chi^2, \tau \rangle = 2$ for the permutation character χ of S_n.

   (d) Show that the *standard* module of S_n (i.e. the complement of the trivial module inside the permutation module) is irreducible.

5. (a) Let H be the trivial subgroup of G, and let ψ be the trivial character of H. Show that $\psi \uparrow G$ is the regular character of G.

   (b) Let H be any subgroup of G. Show that each irreducible representation of G is contained in a representation induced from an irreducible representation of H.

   (c) Let H be an abelian subgroup of G with index n. Show that the degree of each irreducible character χ of G is at most n.