Examples: (Tables copied from last lecture)

1) $G = C_3 = \langle x \rangle$

g_i	e	x	x^2	
$\lvert Z(g_i)\rvert$	3	3	3	
χ_1	1	1	1	$\to W_1$
χ_2	1	ω	ω^2	$\to W_2$
χ_3	1	ω^2	ω	$\to W_3$

$\langle r \mid r^3 = e \rangle$

$\overset{\shortparallel}{C_3} \leq D_3$

Coset reps are e, f.

$W_1 = \langle e + r + r^2 \rangle$

$W_1 \uparrow_{C_3}^{D_3} = \langle e + r + r^2, f + fr + fr^2 \rangle$

$= U_1 \oplus U_2$

2) $G = D_3 = \langle r, f \mid r^3 = e, f^2 = e, fr = r^{-1} f \rangle$

g_i	e	r	f	
$\lvert Cl(g_i)\rvert$	1	2	3	
$\lvert Z(g_i)\rvert$	6	3	2	
χ_1	1	1	1	$\to U_1$
χ_2	1	1	-1	$\to U_2$
χ_3	2	-1	0	$\to U_3$

$W_2 \uparrow_{C_3}^{D_3} = \langle e + \omega r + \omega^2 r^2, f + \omega fr + \omega^2 f r^2 \rangle$

$= U_3$.

Example 3:

$G = A_4$

	g_i	id	$(12)(34)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
	$\lvert Z(g_i)\rvert$	12	4	3	3
ψ_1	χ_1	1	1	1	1
ψ_2	χ_2	1	1	ω	ω^2
ψ_3	χ_3	1	1	ω^2	ω
ψ_4	χ_4	3	-1	0	0

$A_4 \quad S_4$

$\psi_1 = \chi_1 \downarrow_{A_4} = \chi_2 \downarrow_{A_4}$

$\chi_3 \downarrow_{A_4} = \psi_2 + \psi_3$

$\psi_4 = \chi_4 \downarrow_{A_4} = \chi_5 \downarrow_{A_4}$.

4) $G = S_4$:

$V \trianglelefteq A_4 \trianglelefteq S_4$

g_i	e	$(1\,2)$	$(1\,2)(3\,4)$	$(1\,2\,3)$	$(1\,2\,3\,4)$
$\lvert Cl(g_i)\rvert$	1	6	3	8	6
$\lvert Z(g_i)\rvert$	24	4	8	3	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

5.1) Restriction

Defn 1: • Let H be a subgp of G. Then any $\mathbb{C}[G]$-module V can be regarded as a $\mathbb{C}[H]$-module, in which case we denote it by $V\downarrow H$ (or $V\downarrow^G_H$ or $\mathrm{Res}^G_H(V)$ or $\mathrm{Res}_H(V)$) and call it the restriction of V to H.

• If V has character χ, then $V\downarrow H$ has character $\chi\downarrow H$.

Note: χ irred $\not\Rightarrow$ $\chi\downarrow H$ irred.

Prop 1: Let H be a subgp of G and ψ a nonzero char of H. Then there exists an irred. char χ of G s.t. $\langle \chi\downarrow H, \psi\rangle \neq 0$.

Pf: Let $\chi_1, \dots, \chi_k$ be the irred chars of G. Let ρ be the regular character of G. Then $\rho = \sum_i \chi_i(e)\chi_i$.

Now observe that the reg. module $\mathbb{C}[G]$ contains $\mathbb{C}[H]$ is closed under the action of H (so $\mathbb{C}[H]$ is a $\mathbb{C}[H]$-submod of $\mathrm{Res}_H(\mathbb{C}[G])$).

Suppose ψ is irred. Then the corr. $\mathbb{C}[H]$-mod. V is a $\mathbb{C}[H]$-submod of $\mathbb{C}[H]$. Therefore, $\langle \rho\downarrow H, \psi\rangle \neq 0$, so

$$0 \neq \left\langle \sum_i \chi_i(e)\chi_i\downarrow H, \psi\right\rangle_H = \sum_i \chi_i(e)\langle \chi_i\downarrow H, \psi\rangle_H.$$

and so $\langle \chi_i\downarrow H, \psi\rangle \neq 0$ for some i.

We can extend this to reducible ψ by noting that for some constituent ψ_j of ψ, $\exists\, i$ s.t. $\langle \chi_i\downarrow H, \psi_j\rangle \neq 0$. $\square$

Thm 1: (Clifford's Thm) Suppose $H \trianglelefteq G$, χ an irred char of G and $\psi_1, \dots, \psi_m$ the constituents of $\chi\downarrow H$. Then:

i) the ψ_i all have the same degree

ii) $\langle \chi\downarrow H, \psi_i\rangle$ is the same for all ψ_i.

Pf: i) Let V be the $\mathbb{C}[G]$-mod corr to χ and let U be an irred $\mathbb{C}[H]$-submod of $V\downarrow H$. Then $\forall g\in G$, the set $gU = \{gu \mid u\in U\}$ is a subspace of V and $\forall h\in H$, we have $h(gu) = (hg)u = g(g^{-1}hg)u \in gU$ (since $g^{-1}hg\in H$, and U is a $\mathbb{C}[H]$-submod). Thus, gU is also a $\mathbb{C}[H]$-submodule of $V\downarrow H$.

Now suppose W is a submodule of gU. Then $g^{-1}W$ is a submod of U and hence $g^{-1}W$ is a submod of U. Hence $g^{-1}W$ is U or $\{0\}$ (because U is irred) and so W is gU or $\{0\}$, and hence gU is irred. Moreover, since

g is an invertible linear trans, $\dim g\mathcal{U} = \dim \mathcal{U}$.
Finally, note that $\sum_{g\in G} g\mathcal{U}$ is $\mathbb{C}[G]$-submod of V, and so $V = \sum g\mathcal{U}$. Thus, $V = g_1\mathcal{U} \oplus g_2\mathcal{U} \oplus \cdots \oplus g_k\mathcal{U}$.
ii) Exercise.

<u>Prop 2:</u> Let H be a subgp of G, let χ be an irred char of G and let $\psi_1, \dots, \psi_r$ be the irred chars of H. Then

$$\chi \downarrow H = d_1\psi_1 + d_2\psi_2 + \cdots + d_r\psi_r$$

where $\sum d_i^2 \le [G:H] = |G|/|H|$.
Equality holds iff $\chi(g) = 0 \ \forall g \in G - H$.

<u>Pf:</u> • $\sum d_i^2 = \langle \chi\downarrow H, \chi\downarrow H\rangle_H = \frac{1}{|H|}\sum_{h\in H} \chi(h)\overline{\chi(h)}$.

• $1 = \langle \chi, \chi\rangle_G = \frac{1}{|G|}\left(\sum_{h\in H} \chi(h)\overline{\chi(h)} + \sum_{g\in G-H} \chi(g)\overline{\chi(g)}\right)$

$$= \frac{|H|}{|G|}\sum d_i^2 + K$$

where $K > 0$. $\square$

5.2) <u>Induction</u>

Let $H \le G$ (subgp of) and let $g_1, \dots, g_k$ be representatives of the left cosets of H in G. Then given a $\mathbb{C}[H]$-module W, we wish to define a $\mathbb{C}[G]$-module V s.t. $V = g_1W \oplus g_2W \oplus \cdots \oplus g_kW$.

<u>Defn 1:</u> • Let $\mathcal{U}$ be an irred $\mathbb{C}[H]$-mod. Then, viewing $\mathcal{U}$ as a submodule of $\mathbb{C}[H]$ (the regular module), define

$$\mathcal{U}\uparrow G = \{gu \mid g\in G, u\in \mathcal{U}\}.$$

We call $\mathcal{U}\uparrow G$ (or $\mathcal{U}\uparrow_H^G$ or $\mathrm{Ind}_H^G(\mathcal{U})$) the <u>$\mathbb{C}[G]$-module induced from $\mathcal{U}$</u>.
• For a general $\mathbb{C}[H]$-mod $\mathcal{U}$, we have $\mathcal{U} = \bigoplus \mathcal{U}_i$ where the $\mathcal{U}_i$ are irred $\mathbb{C}[H]$-submods. Then $\mathcal{U}\uparrow G = \bigoplus(\mathcal{U}_i\uparrow G)$.
• If $\mathcal{U}$ has character ψ, then the character of $\mathcal{U}\uparrow G$ is denoted $\psi\uparrow G$.

<u>Thm 1:</u> (Frobenius Reciprocity Thm:) $(H\le G)$ Let ψ be a char of H and let χ be a char of G. Then:

$$\langle \psi\uparrow G, \chi\rangle_G = \langle \psi, \chi\downarrow H\rangle_H.$$

Prop 1: Let ψ be a char of H. Then let $\dot{\psi}: G \to \mathbb{C}$ be given by

$$\dot{\psi}(g) = \begin{cases} \psi(g) & \text{if } g \in H \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi \uparrow G\,(g) = \frac{1}{|H|} \sum_{y \in G} \dot{\psi}(ygy^{-1})$.

5.3) Repn Theory of S_n:

[Following Bruce Sagan - The Symmetric Group]

The conjugacy classes of S_n are given by cycle types, e.g. $(1\ 2\ 3)(4\ 5)(6\ 7)(8)$

These are indexed by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n where $\lambda_i \in \mathbb{Z}_{>0}$, $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.
We write $|\lambda| = \sum \lambda_i$, and if $|\lambda| = n$ we write $\lambda \vdash n$
is a partition of

Eg: $n = 4$: Partitions of 4:

$(1,1,1,1)$, $(2,1,1)$, $(2,2)$, $(3,1)$, (4)

$(1)(2)(3)(4)$ $\quad$ $(1\ 2)(3)(4)$ $\quad$ $(1\ 2)(3\ 4)$ $\quad$ $(1\ 2\ 3)(4)$ $\quad$ $(1\ 2\ 3\ 4)$ $\quad$ (cycle types)

Q: How might we find an irred repn corr. to $\lambda \vdash n$?

Let S_λ be the subgp of S_n given by

$$S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_k+1,\ldots,n\}}$$

$$\cong S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_n}$$

Let M^λ be the module associated to $1\uparrow_{S_\lambda}^{S_n}$.
This is not generally irreducible.

BUT, we can order the partitions of n such that

$M^{\lambda^{(1)}}$ is irreducible (call it $S^{\lambda^{(1)}}$)

$M^{\lambda^{(2)}}$ contains copies of $S^{\lambda^{(1)}}$ and a unique copy of a new irreducible submodule (call it $S^{\lambda^{(2)}}$)

$M^{\lambda^{(k)}}$ contains copies of $S^{\lambda^{(i)}}$ for $\lambda^{(i)} < \lambda^{(k)}$ and a unique copy of a new irreducible submodule (call it $S^{\lambda^{(k)}}$).

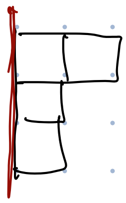
The $S^{\lambda^{(i)}}$ are called Specht modules.
The M^λ are called the permutation modules corr. to λ. (associated to the group S_λ).

Constructing M^λ:

$M^\lambda = \mathbb{C}$-span of $\{\bar{t}_1, \bar{t}_2, \dots, \bar{t}_k\}$ where the $\bar{t}_i$ are distinct λ-tabloids.

To each partition λ, we can associate a Young diagram which is a left-justified array of boxes consisting of λ_1 boxes in the first row, λ_2 boxes in the second, etc.

Eg. $(2,1,1)$:



A tableau is a filling of this Young diagram with the numbers $1, 2, \dots, n$ (each appearing exactly once)

Eg: $\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array}$

We put an equivalence reln on the set of λ-tableau where two tableaux are equivalent whenever all corresponding rows have the same elements.

Eg. $\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array} \sim \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array}$

The equivalence class is called a λ-tabloid, and is denoted without the vertical lines in the Young diagram.

Eg $\begin{array}{cc} \hline 1 & 4 \\ \hline 3 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array}$

Then $\pi \in S_n$ acts on the set of λ-tabloids by permuting elements:

$(1\ 2\ 3) \cdot \begin{array}{cc} \hline 1 & 4 \\ \hline 3 & \\ \cline{1-1} 2 & \end{array} = \begin{array}{cc} \hline 2 & 4 \\ \hline 1 & \\ \cline{1-1} 3 & \end{array}$

Examples:

1) If $\lambda = (n)$, then

$$M^{(n)} = \left\langle \overline{\underline{1\ 2\ 3\ \cdots\ n}} \right\rangle$$

This is a 1-dim module, and S_n acts on this trivially.
This has character $1\uparrow_{S_n}^{S_n}$

2) If $\lambda = (1,1,\dots,1) = (1^n)$, then

$$M^{(1^n)} = \left\langle \begin{array}{c} \overline{\underline{\ }} \\ \overline{\underline{\ }} \\ \vdots \\ \overline{\underline{\ }} \end{array} \right\rangle$$

↳ $n!$ tabloids in all.

This is isomorphic to the regular representation of S_n.

$$1\uparrow_{S_{(1^n)}}^{S_n} = 1\uparrow_{\{e\}}^{S_n} = \rho \text{ (reg. character)}$$

3) If $\lambda = (n-1, 1)$, then

$$M^{\lambda} = \left\langle \begin{array}{c}\overline{\underline{\quad\quad}}\\ \underline{1}\end{array}, \begin{array}{c}\overline{\underline{\quad\quad}}\\ \underline{2}\end{array}, \begin{array}{c}\overline{\underline{\quad\quad}}\\ \underline{3}\end{array}, \dots, \begin{array}{c}\overline{\underline{\quad\quad}}\\ \underline{n}\end{array} \right\rangle$$

This is isomorphic to the defining permutation repn of S_n.
$S_n \circlearrowright \{1, 2, \dots, n\}$

Char: $1\uparrow_{S_{(n-1,1)}}^{S_n}(g) = |\mathrm{fix}(g)|$.

The order on the set of λ needed to extract the Specht modules from the M^{λ} is <u>reverse</u>-<u>lexicographic</u> ordering.

$$(1,1,\dots,1) < (2,1,\dots,1) < (2,2,1,\dots,1) < \quad \cdots \quad < (n-1,1) < (n)$$