From last time:

Throughout, we set $\mathbb{F}=\mathbb{C}$, $A=\mathbb{C}[G]$ for finite gp $G$.

Defn 1: Let $G$ be a finite gp, and let $F(G,\mathbb{C})$ be the space of functions $G \xrightarrow{\text{of}} \mathbb{C}$. Define an inner product on $F(G,\mathbb{C})$ by:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Thm 1: Let $U, V$ be irred $\mathbb{C}[G]$-modules with chars $\chi, \psi$ resp. Then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{if } U \not\cong V. \end{cases}$$ [$\chi=\psi$ in this.]

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HW

Prop 1: Let $\chi$ be the char of the regular repn. Then $\chi(e)=|G|$, and $\chi(g)=0$ for $g\neq e$.

Prop 2: Let $\chi$ be the char of a permutation repn ($S_n$ acting on $\{1,\dots,n\}$) $V=\langle v_1, v_2, \dots, v_n\rangle$
$S_n \curvearrowright V : \sigma\cdot v_i = v_{\sigma(i)}$.

Then $\chi(g)=|fix(g)|$
where $fix(g)=\{i\in\{1,\dots n\} \mid g(i)=i\}$.

Cor 1: Let $\chi$ be the char of a perm. repn. Then $\tilde{\chi}(g)=|fix(g)|-1$ is also a character of $S_n$.

Pf: $U=\langle v_1+v_2+\cdots+v_n\rangle$ is a submodule of the permutation module $V$, and $S_n$ acts trivially on $U$.

## 4.1) The Space of Class Functions on G

Throughout, $G$ is finite, $\mathbb{F}=\mathbb{C}$.

Let $F_c(G,\mathbb{C})$ be the space of class functions on $G$.

Prop 1: $\dim F_c(G,\mathbb{C}) = \#$ of conj. classes of $G$.

Pf: Let $x_1,\dots,x_\ell$ be representatives of the distinct conj. classes of $G$. Then $f_i \in F_c(G,\mathbb{C})$ given by:

$$f_i(x) = \begin{cases} 1 & \text{if } x \text{ is conj. to } x_i \\ 0 & \text{otherwise} \end{cases}$$

is a basis for $F_c(G, \mathbb{C})$. $\square$

Now let $\chi_1, \dots, \chi_k$ be a complete set of distinct irred chars of $G$.

<u>Prop 2</u>: The $\chi_i$ are linearly independent.

<u>Pf</u>: Suppose $\sum_{i=1}^{k} c_i \chi_i = 0$. Then $\forall j$:

$$0 = \langle \underbrace{\sum c_i \chi_i}_{0}, \chi_j \rangle = \sum c_i \langle \chi_i, \chi_j \rangle = c_j$$ $\square$

<u>Lemma 1</u>: Let $f \in F_c(G, \mathbb{C})$ and $\rho$ be an irred. repn. of $G$ on an $n$-dim module $V$ with char $\chi$. Then $\varphi : V \to V$ given by $\varphi = \sum_{g \in G} f(g)\rho(g)$ equals $\lambda 1_V$, where $\lambda = \frac{|G|}{n} \langle f, \overline{\chi} \rangle$.

<u>Pf</u>: We observe $\forall g \in G$:

$$\rho(g)^{-1} \varphi \rho(g) = \rho(g^{-1}) \sum_{x \in G} f(x)\rho(x)\rho(g) = \sum_{x \in G} f(x) \rho(g^{-1}xg)$$

$$= \sum_{y \in G} f(gyg^{-1})\rho(y) = \sum_{y \in G} f(y)\rho(y) = \varphi.$$

Thus, $\varphi \rho(g) = \rho(g)\varphi$, and hence $\varphi$ is a $\mathbb{C}[G]$-hom, and so by Schur's Lemma, $\varphi = \lambda 1_V$ for some $\lambda \in \mathbb{C}$.

In fact:

$$\mathrm{tr}(\varphi) = n\lambda = \sum_{g \in G} f(g)\chi(g) = |G| \langle f, \overline{\chi} \rangle.$$ $\square$

<u>Prop 3</u>: Let $\chi$ be a char of $G$. Then $\overline{\chi}$ is also a char of $G$, and $\overline{\chi}$ is irred iff $\chi$ is irred.

<u>Pf</u>: If $\rho : G \to GL(n, \mathbb{C})$ is a repn, then so is $\sigma(g) = \overline{\rho(g)}$.

Also, $\langle \overline{\chi}, \overline{\chi} \rangle = \langle \chi, \chi \rangle$, so $\langle \overline{\chi}, \overline{\chi} \rangle = 1$ iff $\langle \chi, \chi \rangle = 1$. $\square$

<u>Lemma 2</u>: If $f \in F_c(G, \mathbb{C})$ and $\langle f, \chi_i \rangle = 0$ $\forall i$, then $f = 0$.

<u>Pf</u>: Since $\{\chi_1, \dots, \chi_k\} = \{\overline{\chi}_1, \dots, \overline{\chi}_k\}$, we have $\langle f, \overline{\chi_i} \rangle = 0$ $\forall i$. Let $\varphi_\rho = \sum f(g)\rho(g)$.

If $\rho$ is irred, then by Lemma 1, $\varphi_\rho = 0$.

If $\rho$ is red, then $\varphi_\rho$ acts by $0$ on every irred. submodule, and so again $\varphi_\rho = 0$.

Now let $\rho$ be the regular repn. Then for $h \in G$,

$\varphi_\rho(h) = \sum f(g) gh = 0$.

Since $\{gh \mid g \in G\} = G$ forms a basis for the regular module, we have $f(g) = 0 \ \forall g \in G$. □

<u>Prop 4</u>: The $\chi_i$ span $F_c(G, \mathbb{C})$.
<u>Pf</u>: Let $f \in F_c(G,\mathbb{C})$. Let $\tilde{f} = f - \sum_i \langle f, \chi_i \rangle \chi_i$. Then $\langle \tilde{f}, \chi_j \rangle = \langle f, \chi_j \rangle - \sum_i \langle f, \chi_i \rangle \langle \chi_i, \chi_j \rangle = 0 \quad \forall j$; and so $\tilde{f} = 0$. Thus, $f = \sum \langle f, \chi_i \rangle \chi_i$. □

<u>Cor 1</u>: The $\chi_i$ form a basis for $F_c(G,\mathbb{C})$, and the number of distinct irred chars equals the number of distinct conjugacy classes.

<u>Cor 2</u>: Suppose $g, h \in G$. Then $g$ is conj. to $h$ iff $\chi(g) = \chi(h)$ for all characters $\chi$ of $G$.

<u>Pf</u>: ($\Leftarrow$) Let $f \in F_c(G,\mathbb{C})$ be the function s.t.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is conj to } g \\ 0 & \text{otherwise.} \end{cases}$$

Then since $\chi_i(g) = \chi_i(h)$ for all irred chars $\chi_i$ of $G$, we have $f(h) = f(g) = 1$, so $h$ is conj to $g$. □

<u>Cor 3</u>: For all $g \in G$, $g$ is conj to $g^{-1}$ iff $\chi(g)$ is a real number for all chars $\chi$ of $G$.

## <u>4.2) Character Tables</u>:

Let $\chi_1, \dots, \chi_k$ be the irred. chars of $G$ and let $g_1, \dots, g_k$ be representatives of the distinct conj. classes of $G$.

<u>Defn 1</u>: The $k \times k$ matrix whose $(i,j)$-entry is $\chi_i(g_j)$ is called the <u>character table</u> of $G$.

<u>Note</u>: Since the $\chi_i$ are lin. indep, this matrix is invertible.

<u>Thm 1</u>: i) Row orthogonality:

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\chi_s(g_i^{-1})}{|Z(g_i)|} = \delta_{rs}.$$

($Z(g_i) = \{z \in G \mid z g_i = g_i z\}$, called the centraliser of $g_i$.)

ii) Column orthogonality:

$$\sum_{i=1}^{k} \frac{\chi_i(g_r)\chi_i(g_s^{-1})}{|Z(g_s)|} = \delta_{rs}.$$

Pf: i) $\delta_{rs} = \langle \chi_r, \chi_s \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_r(g) \chi_s(g^{-1}) = \frac{1}{|G|} \sum_{i=1}^{k} \chi_r(g_i) \chi_s(g_i^{-1}) |Cl_G(g_i)|$

where $Cl_G(g_i)$ is the conjugacy class of $g_i$.
By the orbit-stabilizer theorem, $|G| = |Cl_G(g_i)|\,|Z(g_i)|$.

ii) Let $f_s \in F_C(G, \mathbb{C})$ s.t. $f_s(g_r) = \delta_{rs}$. Then
$f_s = \sum_{i=1}^{k} \langle f_s, \chi_i \rangle \chi_i$,

where
$\langle f_s, \chi_i \rangle = \frac{1}{|G|} \sum_{g \in G} f_s(g) \chi_i(g^{-1}) = \frac{1}{|G|} |Cl_G(g_s)| \chi_i(g_s^{-1}) = \frac{\chi_i(g_s^{-1})}{|Z(g_s)|}$

This implies that
$\delta_{rs} = f_s(g_r) = \sum_{i=1}^{k} \frac{\chi_i(g_s^{-1})}{|Z(g_s)|} \chi_i(g_r)$.

$\square$

Examples:

1) $G = C_3 = \langle x \rangle$

| $g_i$ | $e$ | $x$ | $x^2$ |
|---|---|---|---|
| $\|Z(g_i)\|$ | 3 | 3 | 3 |
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | $\omega^2$ | $\omega$ |

$x^{-1} = x^2$.

$\omega = e^{2\pi i/3}$, $\omega^3 = 1$.

$\chi_2, \chi_3$: $\frac{1}{3} + \frac{1}{3}\omega^2 \cdot \overline{\omega} + \frac{1}{3}\omega\overline{\omega^2}$

$= \frac{1 + \omega^2 \cdot \omega^2 + \omega\omega}{3} = \frac{1 + \omega + \omega^2}{3}$

Recall: $|G| = \sum \dim(V_i)^2$

2) $G = D_3 = \langle r, f \mid r^3 = e, f^2 = e, fr = r^{-1} f \rangle$

| $g_i$ | $e$ | $r$ | $f$ |
|---|---|---|---|
| $\|Cl(g_i)\|$ | 1 | 2 | 3 |
| $\|Z(g_i)\|$ | 6 | 3 | 2 |
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ |
| $\chi_3$ | 2 | $x = -1$ | $y = 0$ |

$r \underbrace{f r^{-1}}_{} = r^2 f$

$r^2 f r^{-2} = rf$

$1 + 1 + 2x = 0$
$1 - 1 + 2y = 0$
$1 - 1 + xy = 0$

Prop 1: Suppose $\chi$ is a char of $G$ and $\lambda$ is a linear char of $G$. Then the product $\lambda\chi$ given by $\lambda\chi(g) = \lambda(g)\chi(g)$ is also a char. If $\chi$ is irred, then so is $\lambda\chi$.

Example 3:

$G = A_4$

| $g_i$ / $\lvert Z(g_i)\rvert$ | id / 12 | (12)(34) / 4 | (1 2 3) / 3 | (1 3 2) / 3 |
|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| $\chi_3$ | 1 | 1 | $\omega^2$ | $\omega$ |
| $\chi_4$ | 3 | -1 | 0 | 0 |

where $\omega^3 = 1$, ie $\omega = e^{2\pi i/3}$

1,2 row orthog: $\frac{1}{12} + \frac{x}{4} + \frac{\omega}{3} + \frac{\omega^2}{3}$

$= \frac{1}{3} + \boxed{\frac{-1}{4}} + \frac{x}{4} + \frac{\omega}{3} + \frac{\omega^2}{3}$

4) $G = S_4$:

| $g_i$ / $\lvert Cl(g_i)\rvert$ / $\lvert Z(g_i)\rvert$ | e / 1 / 24 | (1 2) / 6 / 4 | (1 2)(3 4) / 3 / 8 | (1 2 3) / 8 / 3 | (1 2 3 4) / 6 / 4 |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | -1 | 1 | 1 | -1 |
| $\chi_3$ | 2 | 0 | 2 | -1 | 0 |
| $\chi_4$ | 3 | 1 | -1 | 0 | -1 |
| $\chi_5 = \chi_2\chi_4$ | 3 | -1 | -1 | 0 | 1 |
| $\lvert Fix(g)\rvert - 1$: | 3 | 1 | -1 | 0 | -1 |

Prop 2: Let $N$ be a normal subgp of $G$, and let $\tilde{\chi}$ be a char of $G/N$. Let $\chi : G \to \mathbb{C}$ be given by $\chi(g) = \tilde{\chi}(gN)$. Then $\chi$ is a character of $G$, and $\chi$ and $\tilde{\chi}$ have the same degree.

$\chi$ is called the lift (or inflation) of $\tilde{\chi}$ to $G$.