From last time:

Cor 1: Let V be an A-mod with $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an irred submodule. Let W be any irred A-mod. Then $\dim(\mathrm{Hom}_A(V,W))$ and $\dim(\mathrm{Hom}_A(W,V))$ both equal the number of submodules U_i s.t. $U_i \cong W$.

Prop 4: If U is an A-module, then $\dim(\mathrm{Hom}_A(A,U)) = \dim U$.

Pf: Let $d = \dim U$. Let $u_1, u_2, \ldots, u_d$ be a basis for U. Define $\varphi_i : A \to U$ by $\varphi_i(r) = r u_i$. Then $\forall\, r, s \in A$,
$\varphi_i(rs) = (rs)u_i = r(su_i) = r\,\varphi_i(s)$, so $\varphi_i \in \mathrm{Hom}_A(A,U)$.
We wish to show that $\varphi_1, \ldots, \varphi_d$ is a basis for $\mathrm{Hom}_A(A,U)$.
Let $\psi \in \mathrm{Hom}_A(A,U)$. Then:
$\psi(1) = \lambda_1 u_1 + \cdots + \lambda_d u_d$ for some $\lambda_i \in \mathbb{F}$.
Since ψ is an A-homomorphism:
$$\psi(r) = \psi(r1) = r\,\psi(1) = \lambda_1(ru_1) + \lambda_2(ru_2) + \cdots + \lambda_d(ru_d)$$
$$= \lambda_1\varphi_1(r) + \lambda_2\varphi_2(r) + \cdots + \lambda_d\varphi_d(r).$$
Thus, the φ_i span $\mathrm{Hom}_A(A,U)$.
Now suppose $\sum \lambda_i \varphi_i = 0$ for some $\lambda_i \in \mathbb{F}$. Then evaluating both sides at 1 gives us $\sum \lambda_i u_i = 0$, so $\lambda_i = 0\ \forall i$.
So the φ_i are lin. indep, and hence a basis.
Thus, $\dim(\mathrm{Hom}_A(A,U)) = \dim U$. □

Thm 1: Suppose $A = U_1 \oplus \cdots \oplus U_r$, where each U_i is irred. If U is any irred A-module, then the number of A-modules U_i with $U_i \cong U$ is equal to $\dim U$.

Pf: By Prop 4, $\dim U = \dim(\mathrm{Hom}_A(A,U))$, and by Cor 1 to Prop 3, $\dim(\mathrm{Hom}_A(A,U))$ equals the number of submods U_i isom to U. □

Thm 2: Let $V_1, \ldots, V_k$ form a complete set of non-isomorphic irred $\mathbb{F}[G]$-modules. Then $|G| = \sum_{i=1}^{k} (\dim V_i)^2$.

Example: Let G be a gp of order 8. Then the possibilities for dimensions d_i of irred modules are:

1, 1, 1, 1, 1, 1, 1, 1 (abelian)

1, 1, 1, 1, 2 (nonabelian)

Note: For all algebras $\mathbb{F}[G]$, we can define a repn on a 1-dim vect sp V (so $V \cong \mathbb{F}$) given by
$G \times V \to V$
$g\, v \mapsto v.$
This is irred (since V has dim 1), and is called the <u>trivial repn</u>, corr. the <u>trivial module</u>,

3.1) Characters:

For this section, assume $\mathbb{F}$ is alg. closed.

Defn 1: Let A be an alg. with repn ρ on a fin dim. vect sp V. Then the <u>character</u> of ρ (or of V) is the linear map $\chi : A \to \mathbb{F}$ given by
$$\chi(a) = \mathrm{tr}(\rho(a)).$$

For repns of finite gps, we take the char to be $\chi : G \to \mathbb{F}$

Recall: $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
$\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
$\Rightarrow \mathrm{tr}(\underbrace{BAB^{-1}}) = \mathrm{tr}(A)$ ☆

Defn 2: A function $\chi : G \to \mathbb{F}$ satisfying $\chi(sts^{-1}) = \chi(t)$ $\forall s,t \in G$ is called a <u>class function</u>. [constant on conjugacy classes]

Defn 3:
- We say χ is a <u>char of G</u> if χ is the char of some repn of G.
- We say χ is <u>irred</u> if χ is the char of an irred repn.
- A character of a 1-dim repn is called a <u>linear char</u>.

Prop 1: Let χ be the char of a repn ρ of G on V. Then for $s,t \in G$:
i) $\chi(e) = \dim V$ [This is called the degree of V]
ii) $\chi(sts^{-1}) = \chi(t)$.

Prop 2: If V and W are isom. A-modules, then they have the same character.

Idea: If $T : V \to W$ is an isom, $\rho : A \to \mathrm{End}(V)$, $\sigma : A \to \mathrm{End}(W)$, then $\sigma(a) = T \circ \rho(a) \circ T^{-1}$ $\forall a \in A$.

Prop 3: Let $\mathbb{F} = \mathbb{C}$, let χ be a char of G, let $g \in G$ with order m. Then:

i) $\chi(g)$ is a sum of m^{th} roots of unity.

ii) $\chi(g^{-1}) = \overline{\chi(g)}$.

Pf: i) Let $H = \langle g \rangle$. Then H is a cyclic gp and any $\mathbb{C}[G]$-module V is also a $\mathbb{C}[H]$-module. By Maschke's Thm, we can decompose $V = U_1 \oplus \cdots \oplus U_n$ (irred $\mathbb{C}[H]$-mods) and by the corollary to Schur's Lemma, each U_i is 1-dim (because H is abelian). Taking $u_i \in U_i$ (nonzero), we have $u_1, \ldots, u_n$ is a basis $\mathcal{B}$ of V, and $[g]_{\mathcal{B}}$ is a diagonal matrix. Since $g^m = e$, for each i, $gu_i = \omega_i u_i$ where each $\omega_i \in \mathbb{C}$ is an m^{th} root of unity. The trace of $[g]_{\mathcal{B}}$ is the of the ω_i.

ii) $[g^{-1}]_{\mathcal{B}} = \operatorname{diag}\{\omega_1^{-1}, \ldots, \omega_n^{-1}\} = \operatorname{diag}\{\overline{\omega_1}, \overline{\omega_2}, \ldots, \overline{\omega_n}\}$ $\square$

3.2) Inner products of characters

Throughout, we set $\mathbb{F} = \mathbb{C}$, $A = \mathbb{C}[G]$ for finite gp G.

Defn 1: Let G be a finite gp, and let $F(G, \mathbb{C})$ be the space of functions $G \to \mathbb{C}$. Define an inner product on $F(G, \mathbb{C})$ by:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

Prop 1: Let G be a fin gp and let Φ, Ψ be chars of G. Then $\langle \Phi, \Psi \rangle$ is a real number.

Pf:
$$\overline{\langle \Phi, \Psi \rangle} = \langle \Psi, \Phi \rangle \quad \text{(by conj-sym)}$$
$$= \frac{1}{|G|} \sum_{g \in G} \Psi(g) \overline{\Phi(g)}$$
$$= \frac{1}{|G|} \sum_{g \in G} \Psi(g) \Phi(g^{-1})$$
$$= \frac{1}{|G|} \sum_{g \in G} \Psi(g^{-1}) \Phi(g) = \langle \Phi, \Psi \rangle. \quad \square$$

Thm 1: Let U, V be irred $\mathbb{C}[G]$-modules with chars χ, ψ resp. Then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{if } U \not\cong V. \end{cases} \quad [\chi = \psi \text{ in this}]$$

To prove this, we first write a lemma and corollary.

Lemma 1: Let U, V be irred $\mathbb{C}[G]$-mods w/ repns ρ, σ resp. Let $T : V \to U$ be a lin. map. Then $\tilde{T} : V \to U$ given by:

$$\tilde{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g^{-1}) \circ T \circ \sigma(g).$$

is a $\mathbb{C}[G]$-hom. Furthermore,

i) if $U \not\cong V$, $\tilde{T}$ is the zero map

ii) if $U \cong V, \rho = \sigma$, $\tilde{T} = \lambda I_n \quad (\lambda = \frac{1}{n} tr(T))$

Cor 1: Let $\rho(g) = [r_{ij}(g)]$, $\sigma(g) = [s_{ij}(g)]$ and $T = [x_{ij}]$. Then

$$\tilde{T}_{il} = \frac{1}{|G|} \sum_{g \in G} \sum_{j=1}^{m} \sum_{k=1}^{n} r_{ij}(g^{-1}) x_{jk} s_{kl}(g),$$

so if $U \not\cong V$, then

$$\frac{1}{|G|} \sum_{g \in G} r_{ij}(g^{-1}) s_{kl}(g) = 0 \qquad \forall i,j,k,l$$

and if $U \cong V$, $\rho = \sigma$, then:

$$\frac{1}{|G|} \sum_{g \in G} r_{ij}(g^{-1}) r_{kl}(g) = \frac{\delta_{il}\delta_{jk}}{n}.$$

Pf of Thm 1: If $\chi = \psi$:

$$\langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \left(\sum_i r_{ii}(g) \right) \left(\sum_j r_{jj}(g^{-1}) \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ij}}{n} = 1.$$

If $\chi \neq \psi$ (and $U \not\cong V$), then

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^{m} r_{ii}(g) \right) \left(\sum_{j=1}^{n} s_{jj}(g^{-1}) \right) = 0.$$

$\square$

Thm 2: Let V be a $\mathbb{C}[G]$-mod with char ψ s.t $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$, where each V_i is irred. Let W be an irred $\mathbb{C}[G]$-mod with char χ. Then $\langle \psi, \chi \rangle$ is an integer equal to the number of V_i isom to W.

Pf: Let ψ_i be the char of V_i. Then $\psi = \psi_1 + \cdots + \psi_r$,
so $\langle \psi, \chi \rangle = \langle \sum \psi_i, \chi \rangle = \sum \langle \psi_i, \chi \rangle$.
But ψ_i, χ are irred chars, and $\langle \psi_i, \chi \rangle = 1$ if $V_i \cong W$ (and 0 otherwise), so $\langle \psi, \chi \rangle$ equals the number of V_i isom to W. □

Cor 2: Let U and V be $\mathbb{C}[G]$-mods w/ chars χ, ψ resp. Then $U \cong V$ iff $\chi = \psi$.

Pf: If $U \cong V$, then $\chi = \psi$ by Sec. 3.1, Prop 2.
If $U \not\cong V$, then $\chi \neq \psi$ if U and V are irred (since $\langle \chi, \psi \rangle = 0$ in this case, but $\langle \chi, \chi \rangle = 1$). If U and V are not both irred, then there exists an irred mod W with char ξ that appears with different multiplicity in U and V. Then $\langle \chi, \xi \rangle \neq \langle \psi, \xi \rangle$, so $\chi \neq \psi$. □

Thm 3: Let χ be a char of G. Then $\langle \chi, \chi \rangle = 1$ iff χ is irred.

Pf: ($\Leftarrow$:) By Thm 1.
($\Rightarrow$:) Suppose $\langle \chi, \chi \rangle = 1$ but χ is the char of a reducible module V. Then $V = V_1 \oplus \cdots \oplus V_r$ for some irred submods V_i. Let χ_i be the char of V_i. Then $\chi = \chi_1 + \cdots + \chi_r$ and so

$$\langle \chi, \chi \rangle = \left\langle \sum_{i=1}^{r} \chi_i, \sum_{i=1}^{r} \chi_i \right\rangle \geq r > 1,$$

which is a contradiction. □

Example: Char χ of regular $\mathbb{C}[G]$-module.

$\chi(e) = |G|$.

Let $g \in G$ with $g \neq e$.

$\chi(g)$?

permutation matrix

$$\rho(g) = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ \vdots & & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}$$

$g \cdot h \neq h$ for any $h \in G$.
$\hookrightarrow$ all diag entries are 0.

$\chi(g) = 0 \quad \forall g \in G \setminus \{e\}$.