2.1) Semisimple representations

A: assoc alg over $\mathbb{F}$.

Wk 1 Ex 7) Suppose V is an A-mod and W is a submodule. Then V/W is an A-module also.

Pf: Define a map $A \times V/W \to V/W$ by

$$a(\underbrace{v+W}_{\{v+w \mid w \in W\}}) = (av)+W.$$

Suppose $v+W = v'+W$. Then $v-v' \in W$, ie $v-v' = w \in W$. Then $av' = a(v-w) = av - aw$, and $aw \in W$, so $av+W = av'+W$. Thus, we have a well-defined repn.

$(x+W) \in V/W$
$(x+W)+(y+W) = (x+y)+W$

Prop 1: Let $T: V \to W$ be a hom. of A-modules. Then $V/\ker(T) \cong \mathrm{im}(T)$. [isom as A-modules]

~~Prop 2: Let A be finite dimensional.~~

Defn 1: A semisimple (or completely reducible) A-module is a direct sum of irreducible submodules.

Prop 2: Let A be finite dimensional. Then every irreducible A-module V is finite dimensional. [Dimensions are taken as vector space dim for both A and V.]

Pf: Suppose V is an irred A-mod. Let $v \in V$ be nonzero. Then $Av = \{\rho(a)v \mid a \in A\}$ is a submodule of V that is finite-dimensional. But V is irred, and $Av \neq \{0\}$, so $Av = V$. Thus, V is finite dim. $\square$

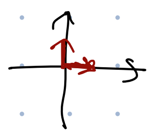
Now, we switch to focussing on the case $A = \mathbb{F}[G]$ for some G.

Defn 2: Let G be a gp, and $A = \mathbb{F}[G]$. Then a representation of G over $\mathbb{F}$ is a repn of A, ie a map $\rho: A \to \mathrm{End}(V)$ s.t. $\rho(ab) = \rho(a)\rho(b)$ and $\rho(1) = \mathrm{Id}_V$.

Note: This restricts to a group homomorphism $\rho: G \to GL(V)$.

Examples: $\mathbb{F} = \mathbb{C}$, $G = D_4 = \langle r, f \mid r^4 = e = f^2, fr = r^{-1}f \rangle$.
↳ Also known as $D_{2(4)}$

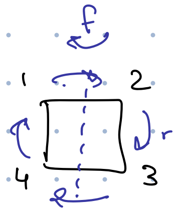
i) $\rho: G \to GL(2, \mathbb{C})$.
$\rho(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\rho(f) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



2) $\rho: G \to GL(n, \mathbb{C})$
$\rho(x) = I_n \quad \forall\ x \in G$. (degree n trivial repn).

3) $\rho: G \to GL(1, \mathbb{C}) \cong \mathbb{C}^\times$
$\rho(r) = 1, \quad \rho(f) = -1$.

4) $\rho: G \to GL(4, \mathbb{C})$
$$\rho(r) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \rho(f) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



__Thm 1: (Maschke's Thm:)__ Let G be a finite gp and let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. Then every finite dimensional $\mathbb{F}[G]$-module is semisimple.

__Non-example:__ $G = C_3 = \langle x \mid x^3 = e \rangle$, $\mathbb{F} = \mathbb{F}_3 \cong \mathbb{Z}_3$, $V = \mathbb{F}^2$.
integers mod 3, ie $\{0,1,2\}$
field of order 3
$\rho: G \to GL(2, \mathbb{F})$.
$$x \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\Rightarrow x^j \mapsto \begin{bmatrix} 1 & j \\ 0 & 1 \end{bmatrix}$$
$U = \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle$ is a submod of V, but it has no complement.

__Pf:__ Let V be an $\mathbb{F}[G]$-module and U a submodule.
[WTS: There exists an $\mathbb{F}[G]$-submodule W of V s.t. $V = U \oplus W$.]
Let W_0 be a subsp of V s.t. $V = U \oplus W_0$. Define $\varphi: V \to V$ by $\varphi(v) = u$ (where $v = u + w$ for $u \in U$, $w \in W_0$). [We want to modify φ to an $\mathbb{F}[G]$-homomorphism.] Let $\psi: V \to V$ be given by:
$$\psi(v) = \frac{1}{|G|} \sum_{g \in G} g \varphi g^{-1}(v).$$

Then ψ is a linear map such that $\operatorname{im}(\psi) = U$. To see that this is a hom of $\mathbb{F}[G]$-modules, we observe that $\forall h \in G$:

$$\psi \circ h(v) = \psi(hv) = \frac{1}{|G|} \sum_{g \in G} g \varphi \underbrace{g^{-1} h}(v) = \frac{1}{|G|} \sum_{k \in G} h k^{-1} \varphi\, k(v)$$
$$= h\, \psi(v).$$

$k = g^{-1} h$
$\Rightarrow g = h k^{-1}$

Thus, $\ker(\psi) =: W$ is a submod, and $V = \operatorname{im}(\psi) \oplus \ker(\psi) = U \oplus W$. $\square$

Example: $G = C_3 = \langle x \rangle$, $V = \langle v_1, v_2, v_3 \rangle_{\mathbb{C}}$, $x \cdot v_i = v_{i+1 \bmod 3}$.

$U = \langle v_1 + v_2 + v_3 \rangle$

$W_0 = \langle v_1, v_2 \rangle$

$\varphi : V \to V$, $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 (v_1 + v_2 + v_3) \mapsto \lambda_3 (v_1 + v_2 + v_3)$.

Exercise: Find ψ and W.

Cor 1: Let G be a fin gp and $\mathbb{F}$ a field s.t. $\operatorname{char}(\mathbb{F}) \nmid |G|$. Then any irred $\mathbb{F}[G]$-module is a direct summand of the regular $\mathbb{F}[G]$-module. [$\mathbb{F}[G]$ acts on $\mathbb{F}[G]$ by left mult.]

Pf: Let U be an irred. $\mathbb{F}[G]$-module. Let $u \in U$ be nonzero. Define an $\mathbb{F}[G]$-hom. $T : \mathbb{F}[G] \to U$ by

$$T\Big(\sum_{g \in G} \lambda_g g\Big) = \sum_{g \in G} \lambda_g g u, \quad \lambda_g \in \mathbb{F}.$$

This is "clearly" an $\mathbb{F}[G]$-mod hom. Since U is irreducible, and $\operatorname{im}(T) \neq 0$ (since, e.g. $T(1) = T(1e) = u \neq 0$), we must have $\operatorname{im}(T) = U$. Then, by Maschke's Thm, $\mathbb{F}[G] = \ker(T) \oplus W$ for some submodule W. Since $\mathbb{F}[G]/\ker(T) \cong W$ and $\mathbb{F}[G]/\ker(T) \cong \operatorname{im}(T) = U$, we have $U \cong W$. $\square$

Cor 2: If G is a fin gp, then there are only finitely many non-isom. irred $\mathbb{F}[G]$-modules (when $\operatorname{char}(\mathbb{F}) \nmid |G|$).

Q: With what multiplicity does each irreducible module appear as a direct summand of $\mathbb{F}[G]$ (the regular $\mathbb{F}[G]$-mod)?

2.2) The space of $\mathbb{F}[G]$-homomorphisms

Throughout this section, let G be a finite gp, and $\mathbb{F}$ be an algebraically closed field s.t. $\operatorname{char}(\mathbb{F}) \nmid |G|$. [Think: $\mathbb{F} = \mathbb{C}$.] Let $A = \mathbb{F}[G]$.

Prop 1: Suppose V, W are irred. A-modules. Then

$$\dim(\operatorname{Hom}_A(V, W)) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Pf: If $V \not\cong W$, then $\operatorname{Hom}_A(V, W) = \{0\}$ by Schur's Lemma. Suppose $V \cong W$. Let $\varphi : V \to W$ be an A-module isomorphism and $\psi \in \operatorname{Hom}_A(V, W)$. Then $\varphi^{-1}\psi$ is an A-mod hom. $V \to V$ and hence $\varphi^{-1}\psi = \lambda \operatorname{Id}_V$ for some $\lambda \in \mathbb{F}$ (by 2nd version of Schur's Lemma). Therefore $\psi = \lambda \varphi$, so $\operatorname{Hom}_A(V, W) = \langle \varphi \rangle$. $\square$

<u>Defn 1:</u> If V is an A-module and U is an irred A-module, we say U is a <u>composition factor</u> of V if V has an A-submodule isomorphic to U.

<u>Prop 2:</u> Let V, W be f.d. A-mods, and suppose $\mathrm{Hom}_A(V,W) \neq \{0\}$. Then V and W have a common composition factor.

<u>Pf:</u> Let $\varphi \in \mathrm{Hom}_A(V,W)$ be nonzero. Then $V = \ker \varphi \oplus U$ for some submod U of V. Let X be an irred submodule of U. Then $\varphi(X) \neq \{0\}$, and by Schur's Lemma, $\mathrm{im}(\varphi) \cong X$ and $\mathrm{im}(\varphi) \subseteq W$!

<u>Prop 3:</u> V_i, W_j irred A-mods. Then:

$$\dim\left(\mathrm{Hom}_A\left(\bigoplus_{i=1}^{r} V_i, \bigoplus_{j=1}^{s} W_j\right)\right) = \sum_{i=1}^{r}\sum_{j=1}^{s} \dim(\mathrm{Hom}_A(V_i, W_j))$$

<u>Cor 1:</u> Let V be an A-mod with $V = U_1 \oplus \cdots \oplus U_r$, where each U_i is an irred submodule. Let W be any irred A-mod. Then $\dim(\mathrm{Hom}_A(V,W))$ and $\dim(\mathrm{Hom}_A(W,V))$ both equal the number of submodules U_i s.t. $U_i \cong W$.