Representation Theory of Finite Groups

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1) Fundamentals

1.1) Algebras

Throughout: Fix a field $\mathbb{F}$.

Defn 0: Recall: A vector space V over $\mathbb{F}$ is a set V together with operations:
i) $+: V \times V \longrightarrow V$
ii) $\cdot: \mathbb{F} \times V \longrightarrow V$
such that
a) V is an abelian gp under $+$
b) $\forall u, v \in V$, $\forall \lambda, \mu \in \mathbb{F}$:
1) $\lambda(u+v) = \lambda u + \lambda v$
2) $(\lambda+\mu) v = \lambda v + \mu v$
3) $(\lambda \mu) v = \lambda(\mu v)$
4) $1_{\mathbb{F}} v = v$.

Defn 1: An associative algebra A over $\mathbb{F}$ is a vect sp A over $\mathbb{F}$ together with a bilinear map $A \times A \longrightarrow A$, $(a,b) \mapsto ab$, such that $(ab)c = a(bc)$.

Defn 2: A unit in an assoc. alg. A is an elt. $1_A \in A$ s.t. $1_A \cdot a = a \cdot 1_A = a$.

Prop 1: If A has a unit, then it must be unique.
Pf: Suppose $1, 1'$ are units in A. Then $1 = 1 \cdot 1' = 1'$. □

Henceforth, by an algebra, we will mean an associative algebra with a unit.

Examples: 1) $A = \mathbb{F}$
2) $A = \mathbb{F}[x_1, x_2, \ldots, x_n]$ (polynomials over $\mathbb{F}$)
3) Let V be a vect sp. over $\mathbb{F}$. Then $A = \mathrm{End}(V)$ is an alg, where $\mathrm{End}(V)$ is the set of all linear maps $V \to V$.
4) The free algebra $A = \mathbb{F}\langle x_1, \ldots, x_n \rangle$ has basis consisting of words in $x_1, \ldots, x_n$, and multiplication given by concatenation: e.g. $(x_1 x_2 x_1) \cdot (x_3 x_1) = x_1 x_2 x_1 x_3 x_1$

5) Let G be a group. The <u>group algebra</u> $A = \mathbb{F}[G]$ has basis indexed by elts in G, and multiplication given by the binary operation on G.
(Elts have the form $\lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_n g_n$, $g_i \in G$, $\lambda_i \in \mathbb{F}$.)

<u>Defn 3:</u> An algebra A is <u>commutative</u> if $ab = ba$ $\forall a, b \in A$.

<u>Defn 4:</u> A <u>homomorphism of algebras</u> $T: A \to B$ is a linear map such that $T(xy) = T(x)T(y)$ $\forall x, y \in A$, and $T(1_A) = 1_B$.

1.2) Representations

<u>Defn 1:</u> A <u>representation</u> of an algebra A on a vect sp V is an algebra homomorphism $\rho: A \to \mathrm{End}(V)$. In this case, we call V a (left) <u>A-module</u>.

<u>Notation:</u> We will often write $\rho(a)v$ as simply av for $a \in A$, $v \in V$. (So we treat $\rho: A \to \mathrm{End}(V)$ as a map $A \times V \to V$.)
Then the hom. property is given by
$(ab)v = a(bv)$.

<u>Examples:</u> 1) A any algebra, $V = 0$, ρ zero map.
2) $V = A$, $\rho: A \to \mathrm{End}(A)$ given by $\rho(a)b = ab$.
This is called the <u>regular repn</u> of A.
3) $A = \mathbb{F}\langle x_1, x_2, \ldots, x_n \rangle$. Then any left A-module is a vect sp V over $\mathbb{F}$ with a collection of arbitrary linear maps $\rho(x_1), \rho(x_2), \ldots, \rho(x_n) \in \mathrm{End}(V)$

<u>Defn 2:</u> A <u>submodule</u> of an A-module V is a subspace $W \leq V$ which is invariant under $\rho(a) \in \mathrm{End}(V)$ $\forall a \in A$.
($\rho(a): W \to W$).
The hom. $\rho_W: A \to \mathrm{End}(W)$ is a <u>subrepresentation</u> of ρ.

<u>Defn 3:</u> An A-module $V \neq 0$ (and its associated repn) is <u>irreducible</u> (or <u>simple</u>) if its only submodules are 0 and V.

<u>Defn 4:</u> • Let $\rho: A \to \mathrm{End}(V)$ and $\sigma: A \to \mathrm{End}(W)$ be two repns of A. Then a <u>homomorphism</u> (or <u>intertwining operator</u>) $\phi: V \to W$ is a linear map which commutes with the action of A, ie $\phi \circ \rho(a) = \sigma(a) \circ \phi$ $\forall a \in A$.

$\phi \circ \rho(a)(v) = \sigma(a) \circ \phi(v)$

$$\begin{array}{ccc} V & \xrightarrow{\rho(a)} & V \\ \phi \downarrow & \circlearrowleft & \downarrow \phi \\ W & \xrightarrow{\sigma(a)} & W \end{array}$$

- A hom. of repns is said to be an <u>isomorphism</u> (or <u>equivalence</u>) of repns if it is an isomorphism of vector spaces.
- The set of all homs of repns. $V_1 \to V_2$ is denoted $\mathrm{Hom}_A(V_1, V_2)$.

<u>Defn 5</u>: Let V_1 and V_2 be A-modules. Then $V_1 \oplus V_2$ is also an A-module, under the map $a(v_1, v_2) \longmapsto (av_1, av_2)$. This is called the <u>direct sum</u> of V_1 and V_2.

<u>Defn 6</u>: A nonzero A-module V is said to be <u>indecomposable</u> if it is not isomorphic to a direct sum of two nonzero modules.

<u>Note</u>: Irreducible $\Rightarrow$ Indecomposable
$\quad\quad\quad\quad\quad\quad\quad\not\Leftarrow$

<u>Example</u>: Let $A = \mathbb{F}[x]$ and $V = \mathbb{F}^2$, and $\rho : A \to \mathrm{End}(V)$ given by:
$$\rho(x) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$
Then $\rho(x)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $W = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ is a submodule.

Hence V is not irreducible.

But if $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \notin W$, then $v_2 \neq 0$, so
$$\rho(x)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ v_2 \end{bmatrix} \neq \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ for any } \lambda \in \mathbb{F}.$$
So $V \neq W \oplus U$ for any submodule U, and thus V is indecomposable.

<u>Prop 1</u>: (<u>Schur's Lemma</u>:) Let V_1, V_2 be modules of an alg. A over a field $\mathbb{F}$ (not necessarily algebraically closed). Let $\phi : V_1 \to V_2$ be a nonzero hom. of repns. Then:

i) If V_1 is irreducible, then ϕ is injective.

ii) If V_2 is irreducible, then ϕ is surjective.

Thus, if V_1 and V_2 are both irreducible, then ϕ is an isomorphism.

Exercise:
- $\ker \phi = \{ v \in V_1 \mid \phi(v) = 0_{V_2} \}$ is a submodule of V_1.
- $\mathrm{im}\, \phi = \{ \phi(v) \in V_2 \mid v \in V_1 \}$ is a submodule of V_2.

Pf: i) We observe $K = \ker \Phi$ is a submod of V_1. Since $\Phi \neq 0$ and V_1 is irred., we must have $K = 0$. Thus, Φ is inj.
ii) The image I of Φ is a submod of V_2. Since $\Phi \neq 0$ and V_2 is irred, we must have $I = V_2$. Thus, Φ is surj. □

Cor 1: (Schur's Lemma for alg. closed fields:) Let V be a fin. dim. irred A-module over an alg. closed field $\mathbb{F}$, and let $\Phi : V \to V$ be an intertwining op. Then $\Phi = \lambda \, Id_V$ for some $\lambda \in \mathbb{F}$.

Pf: Let λ be an eigenvalue of Φ (which exists because $\mathbb{F}$ is alg. closed). Then $\Phi - \lambda Id$ is also an intertwining op which is not an isom (since its det is 0). Then by Prop 1, the operator is the 0 map. Thus, $\Phi = \lambda \, Id$. □

Cor 2: Let A be a commutative alg. over an alg. closed $\mathbb{F}$. Then every irred fin. dim A-module V is 1-dimensional.

Pf: Let V be irred. Then $\forall a \in A$, $\rho(a) : V \to V$ is an intertwining op. Thus, by Schur's Lemma, $\rho(a)$ is a scalar operator $\forall a \in A$. Hence, every subsp of V is a submod. So if V is irred, it cannot have any subspaces, and hence $\dim V = 1$. □