Math 405: Lie Algebras
Exercise Set 9

Throughout this homework, let Φ denote a root system in a Euclidean space E, with Weyl group W.

1. Let E' be a supspace of E. Show that if a reflection leaves E' invariant, then either $\alpha \in E'$ or $E' \subset P_\alpha$.

2. Let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$.

   (a) Show that $\Phi^\vee$ is a root system in E whose Weyl group is isomorphic to W.

   (b) Prove that if Φ is irreducible, so is $\Phi^\vee$.

   (c) Show that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$.

   (d) Show that if Φ has all roots of equal length, so does $\Phi^\vee$, and if Φ has roots of two different lengths, so does $\Phi^\vee$, but in that case long roots in Φ become short roots in $\Phi^\vee$ and vice versa.

3. (a) Show that the order of $\sigma_\alpha \sigma_\beta$ in W is k when the angle between α and β is $\theta = {}^{\pi}\!/_{k}$ or $\theta = \pi - {}^{\pi}\!/_{k}$. (*Hint*: $\sigma_\alpha \sigma_\beta$ produces a rotation of 2θ.)

   (b) Use the result of the previous problem to classify all the Weyl groups of rank 2 as dihedral groups. (Note that there are 4 types of rank 2 root systems.)

[**Additional problems may be added later this week.**]